NUMERICAL DATA ON S_3-SEXTIC FIELDS

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Abstract. We discuss some numerical data in conjuction with our S_3 sextic fields paper.

I computed all negative discriminants of S_3-sextic fields between 0 and 10^{16}.

First, I downloaded the latest version of Karim Belabas's cubic program and after a fair amount of effort, I succeeded in getting it to run (and to compute a list of negative cubic field discriminants up to 10^9).

Unfortunately I could not succeed in getting my PARI/GP program to compute the discriminants of the S_3 sextic extensions, so I wrote a program in Java to accomplish this instead.

I observed also that my previous PARI/GP program, which produced data that seemed to be good, was in fact wrong. (Note that the estimates start off being too low and then switch over to being too high – if I recall correctly.) If you have the original code, you will observe that it does not handle ramification at 2 correctly.

My program computed the following data:

```
Discriminants < 100000000: 91
Discriminants < 150000000: 113
Discriminants < 200000000: 125
Discriminants < 250000000: 130
Discriminants < 300000000: 137
Discriminants < 400000000: 162
Discriminants < 500000000: 182
Discriminants < 600000000: 192
Discriminants < 800000000: 217
Discriminants < 1000000000: 235
Discriminants < 1500000000: 265
Discriminants < 2000000000: 301
Discriminants < 2500000000: 329
Discriminants < 3000000000: 355
Discriminants < 4000000000: 391
Discriminants < 5000000000: 422
Discriminants < 6000000000: 450
Discriminants < 8000000000: 498
Discriminants < 10000000000: 537
Discriminants < 15000000000: 611
Discriminants < 20000000000: 685
Discriminants < 25000000000: 748
Discriminants < 30000000000: 792
Discriminants < 40000000000: 874
Discriminants < 50000000000: 963
Discriminants < 60000000000: 1021
```

```
Discriminants < 80000000000: 1142
Discriminants < 100000000000: 1232
Discriminants < 150000000000: 1430
Discriminants < 200000000000: 1580
Discriminants < 250000000000: 1730
Discriminants < 300000000000: 1843
Discriminants < 400000000000: 2019
Discriminants < 500000000000: 2189
Discriminants < 600000000000: 2326
Discriminants < 800000000000: 2594
Discriminants < 1000000000000: 2809
Discriminants < 1500000000000: 3258
Discriminants < 2000000000000: 3625
Discriminants < 2500000000000: 3914
Discriminants < 3000000000000: 4179
Discriminants < 4000000000000: 4607
Discriminants < 5000000000000: 4976
Discriminants < 6000000000000: 5305
Discriminants < 8000000000000: 5847
Discriminants < 10000000000000: 6315
Discriminants < 15000000000000: 7274
Discriminants < 20000000000000: 8050
Discriminants < 25000000000000: 8724
Discriminants < 30000000000000: 9293
Discriminants < 40000000000000: 10292
Discriminants < 50000000000000: 11116
Discriminants < 60000000000000: 11813
Discriminants < 80000000000000: 13047
Discriminants < 100000000000000: 14121
Discriminants < 150000000000000: 16212
Discriminants < 200000000000000: 17888
Discriminants < 250000000000000: 19274
Discriminants < 300000000000000: 20539
Discriminants < 400000000000000: 22710
Discriminants < 500000000000000: 24584
Discriminants < 600000000000000: 26243
Discriminants < 800000000000000: 28973
Discriminants < 1000000000000000: 31276
Discriminants < 1500000000000000: 35954
Discriminants < 2000000000000000: 39589
Discriminants < 2500000000000000: 42785
Discriminants < 3000000000000000: 45530
Discriminants < 4000000000000000: 50325
Discriminants < 5000000000000000: 54260
Discriminants < 6000000000000000: 57788
Discriminants < 8000000000000000: 63849
Discriminants < 10000000000000000: 68972
```

```
Discriminants < 15000000000000000: 79220
Discriminants < 20000000000000000: 87462
Discriminants < 25000000000000000: 94376
Discriminants < 30000000000000000: 100396
Discriminants < 40000000000000000: 110828
Discriminants < 50000000000000000: 119761
Discriminants < 60000000000000000: 127572
Discriminants < 80000000000000000: 140706
Discriminants < 100000000000000000: 151877
Discriminants < 150000000000000000: 174418
Discriminants < 200000000000000000: 192486
Discriminants < 250000000000000000: 207926
Discriminants < 300000000000000000: 221431
Discriminants < 400000000000000000: 244033
Discriminants < 500000000000000000: 263268
Discriminants < 600000000000000000: 280084
Discriminants < 800000000000000000: 308937
Discriminants < 1000000000000000000: 333398
```

I then used PARI/GP to compute the following expected counts. I have added real counts from the above tables in brackets to the right. The second figure in square brackets is explained after the data.

```
X = 1 * 10^10; predicted = 594.50313644675447191186769 21
X = 2 * 10^10; predicted = 759.00464796862086435271666 37
X = 5 * 10^10; predicted = 1047.2471020877481526953314 36
X = 1 * 10^11; predicted = 1335.0508676997039460585010 22 [1232]
X = 2 * 10^11; predicted = 1700.9736438522840065853331 93 [1580]
X = 5 * 10^11; predicted = 2341.0300952563240399862073 20 [2189]
X = 1 * 10^12; predicted = 2979.0907409917359706068081 77 [2809] [0.079]
X = 2 * 10^12; predicted = 3789.2768848065999343198751 58 [3625] [0.063]
X = 5 * 10^12; predicted = 5204.3635693144160676378331 78 [4976] [0.068]
X = 1 * 10^13; predicted = 6613.1586094513584020516860 65 [6315] [0.073]
X = 2 * 10^13; predicted = 8400.0345547284049892282781 38 [8050] [0.070]
X = 5 * 10^13; predicted = 11517.233377583176412666761 78 [11116] [0.062]
X = 1 * 10^14; predicted = 14617.101315656903968339827 04 [14121] [0.064]
X = 2 * 10^14; predicted = 18545.259710404342637691932 07 [17888] [0.070]
X = 5 * 10^14; predicted = 25390.894200462333608429709 93 [24584] [0.067]
X = 1 * 10^15; predicted = 32192.032232100330956288818 64 [31276] [0.062]
X = 2 * 10^15; predicted = 40803.715059671452391533866 60 [39589] [0.068]
X = 5 * 10^15; predicted = 55798.316926101810943068397 54 [54260] [0.067]
X = 1 * 10^16; predicted = 70683.484025803762116031169 70 [68972] [0.062]
X = 2 * 10^16; predicted = 89518.730062861781499627142 67 [87462] [0.061]
X = 5 * 10^16; predicted = 122290.21685257071074671764 66 [119761] [0.058]
X = 1 * 10^17; predicted = 154800.20835103065783626787 32 [151877] [0.055]
X = 2 * 10^17; predicted = 195913.99537391141950439001 80 [192486] [0.054]
X = 5 * 10^17; predicted = 267402.56672285734454415470 71 [263268] [0.050]
X = 1 * 10^18; predicted = 338278.99530271796817254074 46 [333398] [0.049]
```

There is a small, but consistent and unmistakable, discrepancy.

The second term in brackets is a little bit odd. It is the difference between predicted and actual, divided by $X^{5/18}$. It seems to hold steady for awhile (which worried me when I had computed only through 10^{16}, and then slowly gets better. The slowly getting better is what I might expect, but it also makes me question whether there may be some mistake in my paper.

One way of fixing this is the following (as discussed at Princeton). No field can be totally ramified at any prime $> X^{1/4}$. This suggests multiplying the main term by a factor of

$$\prod_{p > X^{1/4}} \frac{1 + p^{-1}}{1 + p^{-1} + p^{-4/3}} = 1 + \sum_{p > X^{1/4}} -p^{-4/3} + O(p^{7/3}) \sim 1 - \int_{X^{1/4}}^{\infty} \frac{t^{-4/3}}{\log t}. \tag{0.1}$$

We approximate the latter integral with $1 - \int_{X^{1/4}}^{\infty} \frac{t^{-4/3}}{\log X}$, which is not *too* far off when $X > 10^{10}$, say. (Later, we numerically evaluate this integral more accurately.) Doing this suggests that we multiply the main term by a factor of

$$1 - 12 \frac{X^{-1/12}}{\log X}. \tag{0.2}$$

```
X = 1 * 10^10; predicted = 532.99061356253418186245953O1
X = 2 * 10^10; predicted = 687.99123483762909041782550 42
X = 5 * 10^10; predicted = 961.27370830546535043255569 14
X = 1 * 10^11; predicted = 1235.6086377864703138765127 53 [1232]
X = 2 * 10^11; predicted = 1585.8663057345018212853843 13 [1580]
X = 5 * 10^11; predicted = 2201.2135994688170651750367 92 [2189]
X = 1 * 10^12; predicted = 2816.9910102105976145611620 18 [2809]
X = 2 * 10^12; predicted = 3601.2241901107196771067128 75 [3625]
X = 5 * 10^12; predicted = 4975.3097470496830361509901 10 [4976]
X = 1 * 10^13; predicted = 6347.0737353397113892666613 35 [6315]
X = 2 * 10^13; predicted = 8090.7659993844011159660437 83 [8050]
X = 5 * 10^13; predicted = 11139.645427757892912534723 45 [11116]
X = 1 * 10^14; predicted = 14177.726152198187610844054 46 [14121]
X = 2 * 10^14; predicted = 18033.750190475626882216762 54 [17888]
X = 5 * 10^14; predicted = 24765.117863088847787020133 85 [24584]
X = 1 * 10^15; predicted = 31462.789212736617653620904 67 [31276]
X = 2 * 10^15; predicted = 39953.555638273858144141825 06 [39589]
X = 5 * 10^15; predicted = 54756.395710966271203921661 30 [54260]
X = 1 * 10^16; predicted = 69467.736044945224445688823 51 [68972]
```

We observe two things. In the first place, our replacement of $\log t$ by $\log X$ above made the resulting integrals larger, and so the true "predicted" values above should be a bit bigger. Secondly, if we correct our main term then it seems we should similarly correct the secondary term to reflect the fact that no prime $> X^{1/4}$ can be totally ramified. This suggests that we should multiply the secondary term by a factor

$$\prod_{p > X^{1/4}} \frac{1 + p^{-2/3} + p^{-1} + p^{-4/3}}{1 + p^{-2/3} + p^{-1} + p^{-4/3} + p^{-13/9}}. \tag{0.3}$$

This product is

$$\prod_{p>X^{1/4}} \left(1 - p^{-13/9} + O(p^{-19/9})\right) \sim 1 - 9\frac{X^{-1/9}}{\log X}, \tag{0.4}$$

by an identical calculation above. The secondary term is negative, and so this correction factor is positive, and if we incorporate it then we get the following data:

```
X = 1 * 10^10; predicted = 539.3348888672780116958292062
X = 2 * 10^10; predicted = 694.9043379512001267376677016
X = 5 * 10^10; predicted = 969.0278735428965709376037977
X = 1 * 10^11; predicted = 1244.074203828193746129683763
X = 2 * 10^11; predicted = 1595.115466442181319136700778
X = 5 * 10^11; predicted = 2211.622289140147632027707859
X = 1 * 10^12; predicted = 2828.381282543641537615369817
X = 2 * 10^12; predicted = 3613.696461214772000491201278
X = 5 * 10^12; predicted = 4989.384534894444882384835980
X = 1 * 10^13; predicted = 6362.506320005980119991843098
X = 2 * 10^13; predicted = 8107.696445959674790584649801
X = 5 * 10^13; predicted = 11158.79632414378275388043465
X = 1 * 10^14; predicted = 14198.76009320666208975448764
X = 2 * 10^14; predicted = 18056.86301572091960538566003
X = 5 * 10^14; predicted = 24791.31517821996314444431528
X = 1 * 10^15; predicted = 31491.60457573023266491674764
X = 2 * 10^15; predicted = 39985.26345602641165609365417
X = 5 * 10^15; predicted = 54792.39875278980860693946367
X = 1 * 10^16; predicted = 69507.38776559987338214738398
```

The addition turns out to make very little difference.

To try and improve our estimate, we make one further estimation. If a field K has S_3-sextic closure with absolute discriminant $< X$, then no prime greater than $X^{1/4}$ can totally ramify in K. But in fact something slightly stronger is true. Suppose that $n = ap^2$ is the discriminant of a field being counted. In particular note that a cannot be 1 or 2. (If n is negative, then we can have $a = -3$, e.g. $x^3 - x^2 + 6x - 12$.)

Then the discriminant of the Galois closure is $\widetilde{a}p^4$ (where $\widetilde{a}$ does not depend only on a), and we know that $\widetilde{a} \geq 9$. Therefore, no prime greater than $9^{-1/4}X^{1/4}$ can totally ramify in K, and we might replace $\frac{X^{-1/12}}{\log X}$ above with $\frac{(X/9)^{-1/12}}{\log X - \log 9}$, and modify the second term similarly.

We obtain the following data:

```
X = 1 * 10^12; predicted = 2783.397504523112067902031609
X = 2 * 10^12; predicted = 3561.704812372648490979578533 [3625]
X = 5 * 10^12; predicted = 4926.361063973435720398944014 [4976]
X = 1 * 10^13; predicted = 6289.553769729025357013244877 [6315]
X = 2 * 10^13; predicted = 8023.199647078455507293596393 [8050]
X = 5 * 10^13; predicted = 11056.09881939148330850432134 [11116]
X = 1 * 10^14; predicted = 14079.65737371099172775879401 [14121]
X = 2 * 10^14; predicted = 17918.66302338746812846229167 [17888]
X = 5 * 10^14; predicted = 24622.96437547107682953454138 [24584]
X = 1 * 10^15; predicted = 31296.04115214398959785109885 [31276]
X = 2 * 10^15; predicted = 39757.98634743359929168577637 [39589]
```

```
X = 5 * 10^15; predicted = 54514.98821433822401521777975 [54260]
X = 1 * 10^16; predicted = 69184.67423163734637517800397 [68972]
```

Finally, the following data was generated by numerically evaluating the integral of $\frac{t^a}{\log t}$ (by dividing the region $[X^{1/4}, \infty)$ into 10,000 intervals $[Y, 1.01Y]$, discarding the tail which is tiny, approximating $\log t \sim \log Y$ on each interval, and evaluating each integral and adding the results. If we replace 1.01 with, say, 1.005, this changes each result by a maximum of 0.3 or so. If we go further into the tail the change is negligible (around 10^{10}).

```
X = 1 * 10^12; predicted = 2836.408462864091840230620276 [2809]
X = 2 * 10^12; predicted = 3622.290158310473722675473596 [3625]
X = 5 * 10^12; predicted = 4998.731182837070309589775294 [4976]
X = 1 * 10^13; predicted = 6372.410113670674140375344651 [6315]
X = 2 * 10^13; predicted = 8118.128732490010172265140375 [8050]
X = 5 * 10^13; predicted = 11169.84611601533254334089432 [11116]
X = 1 * 10^14; predicted = 14210.17895862219259530440892 [14121]
X = 2 * 10^14; predicted = 18068.52569450801977100773961 [17888]
X = 5 * 10^14; predicted = 24803.02619137144663729058938 [24584]
X = 1 * 10^15; predicted = 31503.06620819541751342495342 [31276]
X = 2 * 10^15; predicted = 39996.14368181792158787546814 [39589]
X = 5 * 10^15; predicted = 54801.83232670346644456614047 [54260]
X = 1 * 10^16; predicted = 69515.05419816393115897364505 [68972]
```

The following data is perhaps the most "pure": it takes the product from $p > X^{1/4}$ (rather than $(X/9)^{1/4}$ and otherwise incorporates all the suggestions above.

```
X = 1 * 10^10; predicted = 555.3054625479044689828620612
X = 2 * 10^10; predicted = 713.1143752645227603305077490
X = 5 * 10^10; predicted = 990.7161494615694027770883989
X = 1 * 10^11; predicted = 1268.852225009104774729470729
X = 2 * 10^11; predicted = 1623.445571406026419934102500
X = 5 * 10^11; predicted = 2245.479999402051380472486399
X = 1 * 10^12; predicted = 2867.158355212011386331935925 [2809]
X = 2 * 10^12; predicted = 3658.137952678139863404776160 [3625]
X = 5 * 10^12; predicted = 5042.656681517150276689360533 [4976]
X = 1 * 10^13; predicted = 6423.651068528423212850082928 [6315]
X = 2 * 10^13; predicted = 8177.919462937101265288679331 [8050]
X = 5 * 10^13; predicted = 11243.19583131460305431869252 [11116]
X = 1 * 10^14; predicted = 14295.81842446922398168218415 [14121]
X = 2 * 10^14; predicted = 18168.53874102988965186901354 [17888]
X = 5 * 10^14; predicted = 24925.85271820055379297794168 [24584]
X = 1 * 10^15; predicted = 31646.58694177200602837515339 [31276]
X = 2 * 10^15; predicted = 40163.88370401186321952365744 [39589]
X = 5 * 10^15; predicted = 55008.04229562610753846871978 [54260]
X = 1 * 10^16; predicted = 69756.18627528715720029091936 [68972]
X = 2 * 10^16; predicted = 88430.51511716406686510920679 [87462]
X = 5 * 10^16; predicted = 120945.3922816623920360980536 [119761]
X = 1 * 10^17; predicted = 153221.5571329228232269737078 [151877]
X = 2 * 10^17; predicted = 194060.6341399497616631595404 [192486]
X = 5 * 10^17; predicted = 265110.9332131825898390446903 [263268]
```

```
X = 1 * 10^18; predicted = 335587.823505226215553298782 3 [333398]
```

1. Summary of numerical data

X	Count	I	IV	V
$1 \cdot 10^{12}$	2809	2979	2867	2766
$1 \cdot 10^{13}$	6315	6613	6423	6256
$5 \cdot 10^{13}$	11116	11517	11243	11004
$1 \cdot 10^{14}$	14121	14617	14295	14017
$2 \cdot 10^{14}$	17888	18545	18168	17844
$5 \cdot 10^{14}$	24854	25390	24925	24528
$1 \cdot 10^{15}$	31276	32192	31646	31182
$2 \cdot 10^{15}$	39589	40803	40163	39622
$5 \cdot 10^{15}$	54260	55798	55008	54343
$1 \cdot 10^{16}$	68972	70683	69756	68979
$2 \cdot 10^{16}$	87462	89519	88430	87523
$5 \cdot 10^{16}$	119761	122290	120945	119830
$1 \cdot 10^{17}$	151877	154800	153221	151917
$2 \cdot 10^{17}$	192486	195914	194060	192536
$5 \cdot 10^{17}$	263268	267402	265110	263235
$1 \cdot 10^{18}$	333398	338278	335587	333393

- I: Original heuristic with terms of order $X^{1/3}$ and $X^{5/18}$.
- IV: Two main terms, truncated, each taken with "correct truncation" after $X^{1/4}$.
- V: Truncate products at $(X/310)^{1/4}$ instead of $X^{1/4}$. This parameter was chosen fairly arbitrarily, to get a close numerical match to the data. Note however that it seems to go from too low to too high.

2. Positive discriminants

```
Discriminants < 1000000000000: 690
Discriminants < 1500000000000: 803
Discriminants < 2000000000000: 895
Discriminants < 2500000000000: 966
Discriminants < 3000000000000: 1046
Discriminants < 4000000000000: 1181
Discriminants < 5000000000000: 1280
Discriminants < 6000000000000: 1363
Discriminants < 8000000000000: 1511
Discriminants < 10000000000000: 1650
Discriminants < 15000000000000: 1909
Discriminants < 20000000000000: 2123
Discriminants < 25000000000000: 2305
Discriminants < 30000000000000: 2472
Discriminants < 40000000000000: 2752
Discriminants < 50000000000000: 2984
Discriminants < 60000000000000: 3185
Discriminants < 80000000000000: 3561
```

```
Discriminants < 100000000000000: 3848
Discriminants < 150000000000000: 4486
Discriminants < 200000000000000: 4981
Discriminants < 250000000000000: 5412
Discriminants < 300000000000000: 5777
Discriminants < 400000000000000: 6414
Discriminants < 500000000000000: 6948
Discriminants < 600000000000000: 7403
Discriminants < 800000000000000: 8179
Discriminants < 1000000000000000: 8867
Discriminants < 1500000000000000: 10202
Discriminants < 2000000000000000: 11324
Discriminants < 2500000000000000: 12259
Discriminants < 3000000000000000: 13086
Discriminants < 4000000000000000: 14493
Discriminants < 5000000000000000: 15740
Discriminants < 6000000000000000: 16753
Discriminants < 8000000000000000: 18546
Discriminants < 10000000000000000: 20062
Discriminants < 15000000000000000: 23180
Discriminants < 20000000000000000: 25578
Discriminants < 25000000000000000: 27725
Discriminants < 30000000000000000: 29498
Discriminants < 40000000000000000: 32628
Discriminants < 50000000000000000: 35337
Discriminants < 60000000000000000: 37637
Discriminants < 80000000000000000: 41651
Discriminants < 100000000000000000: 45054
Discriminants < 150000000000000000: 51914
Discriminants < 200000000000000000: 57403
Discriminants < 250000000000000000: 61978
Discriminants < 300000000000000000: 66076
Discriminants < 400000000000000000: 73095
Discriminants < 500000000000000000: 78905
Discriminants < 600000000000000000: 84049
Discriminants < 800000000000000000: 92878
Discriminants < 1000000000000000000: 100335
```

3. Summary of numerical data (positive discs)

X	Count	I	IV
$1 \cdot 10^{12}$	690	756	721
$1 \cdot 10^{13}$	1650	1762	1702
$5 \cdot 10^{13}$	2984	3154	3066
$1 \cdot 10^{14}$	3848	4045	3942
$2 \cdot 10^{14}$	4981	5182	5061
$5 \cdot 10^{14}$	8179	7181	7030
$1 \cdot 10^{15}$	8867	9181	9004
$2 \cdot 10^{15}$	11324	11729	11523
$5 \cdot 10^{15}$	15740	16197	15940
$1 \cdot 10^{16}$	20062	20658	20357
$2 \cdot 10^{16}$	25578	26333	25979
$5 \cdot 10^{16}$	35337	36260	35822
$1 \cdot 10^{17}$	45054	46159	45644
$2 \cdot 10^{17}$	57403	58730	58125
$5 \cdot 10^{17}$	78905	80690	79941
$1 \cdot 10^{18}$	100335	102555	101674

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