Midterm Exam 1 - Math 141, Frank Thorne (thorne@math.sc.edu)

Wednesday, September 27, 2023

Please work without books, notes, calculators, phones, or assistance from others. If you have any questions, ask. Please do your work on separate paper and turn that in.

20 points each for Questions 1 and 5, 15 points each for the others.

GOOD LUCK!

(1) Give the definition of the *derivative* of a function $f(x)$ at the point $x = a$. (Please give the algebraic definition, using an equation.)

Draw a picture and explain why your equation gives the slope of the tangent line to the graph of $f(x)$ at $x = a$.

(2) Evaluate the limit

$$\lim_{h \to 0} \frac{\sqrt{5h+4} - 2}{h}.$$

(3) Evaluate the limit

$$\lim_{x \to -\pi} \sqrt{x+4} \cos(x + \pi).$$

(4) (a) Graph

$$f(x) = \begin{cases} 1 - x^2, & x \neq 1, \\ 2, & x = 1. \end{cases}$$

(b) Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

(c) Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

(5) Find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

$$g(x) = \frac{x}{x-2}, \quad (3, 3)$$

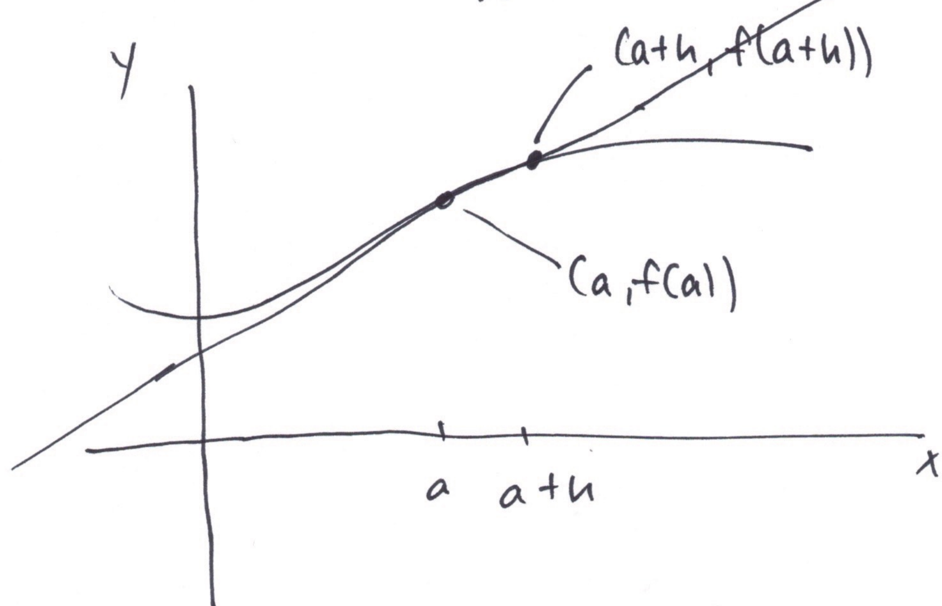
Instructions: For this problem, use the definition of the derivative (or, if you prefer, the 'alternative formula' from Section 3.2 if you know it). Do not use any differentiation rules from later chapters.

(6) Find the derivative of the function

$$f(s) = \frac{\sqrt{s} - 1}{\sqrt{s} + 1}.$$

Instructions: You may solve this problem by any means you know.

1. We have $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$



The slope of the line between $(a, f(a))$ and $(a+h, f(a+h))$ is

$$\frac{\text{rise}}{\text{run}} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{a}.$$

This is the quantity inside the limit. As $h \to 0$, the two points approach each other and hence this line approaches the tangent line, so the slope is the limit of this quantity as $h \to 0$.

2.

$$\lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h} = \lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h} \cdot \frac{\sqrt{5h+4}+2}{\sqrt{5h+4}+2}$$

$$= \lim_{h\to 0} \frac{(5h+4)-4}{h(\sqrt{5h+4}+2)}$$

$$= \lim_{h\to 0} \frac{5h}{h(\sqrt{5h+4}+2)}$$

$$= \lim_{h\to 0} \frac{5}{\sqrt{5h+4}+2}$$

$$= \frac{5}{\sqrt{5\cdot 0+4}+2} = \frac{5}{2+2} = \frac{5}{4}.$$

3. This function is continuous, so we can just plug in $x = -\pi$:

$$\lim_{x\to -\pi} \sqrt{x+4}\cos(x+\pi) = \sqrt{-\pi+4}\cos(-\pi+\pi)$$

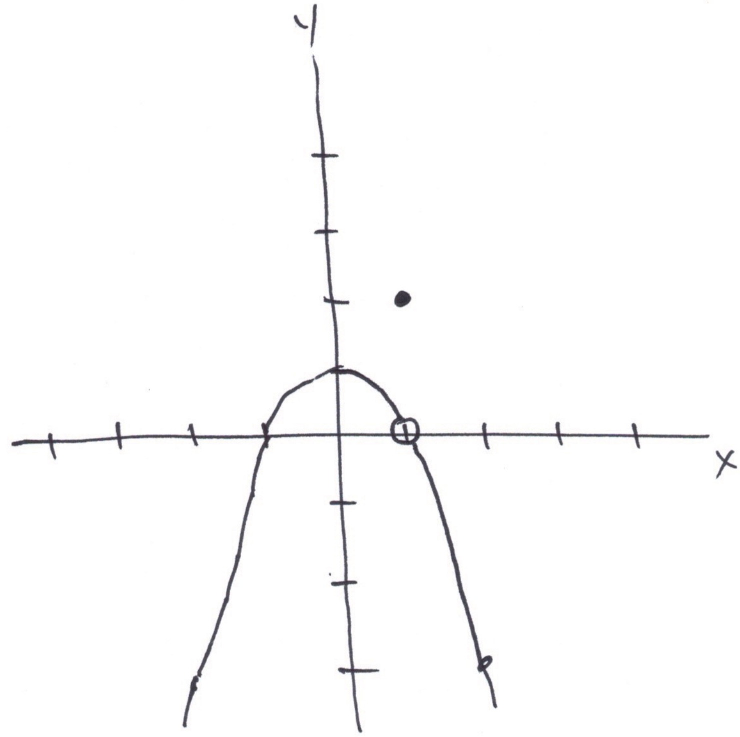
$$= \sqrt{4-\pi}\cos(0)$$

$$= \sqrt{4-\pi}.$$

4.

$$f(x) = \begin{cases} 1 - x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$

(a)



(b) $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 0$,

because this is what $f(x)$ approaches as $x \to 1$ from the left or the right.

(c) Yes, $\lim_{x \to 1} f(x)$ exists and equals 0, because this is the common value of $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

[Or: Because $f(x)$ approaches 0 as x approaches 1 from either direction.]

5.

The slope at $(3, 3)$ is

$$g'(3) = \lim_{h \to 0} \frac{\dfrac{3+h}{3+h-2} - \dfrac{3}{3-2}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{3+h}{1+h} - 3}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3+h}{1+h} - \frac{3(1+h)}{1+h}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3+h-3-3h}{1+h}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-2h}{1+h}}{h} = \lim_{h \to 0} \frac{-2h}{h(1+h)} = \lim_{h \to 0} \frac{-2}{1+h}$$

$$= \frac{-2}{1} = -2.$$

The tangent line is $y - 3 = -2(x-3)$

$$y - 3 = -2x + 6$$

$$\boxed{y = -2x + 9.}$$

6. $$\frac{df}{ds} = \frac{(\sqrt{s}+1)\frac{d}{ds}(\sqrt{s}-1) - (\sqrt{s}-1)\frac{d}{ds}(\sqrt{s}+1)}{(\sqrt{s}+1)^2}$$

$$= \frac{(\sqrt{s}+1)\cdot\frac{1}{2\sqrt{s}} - (\sqrt{s}-1)\cdot\frac{1}{2\sqrt{s}}}{(\sqrt{s}+1)^2}$$

$$= \frac{(\sqrt{s}+1-\sqrt{s}+1)\cdot\frac{1}{2\sqrt{s}}}{(\sqrt{s}+1)^2}$$

$$= \frac{\frac{2}{2\sqrt{s}}}{(\sqrt{s}+1)^2} = \frac{1}{\sqrt{s}(\sqrt{s}+1)^2}.$$