Midterm Exam 2 - Math 141, Frank Thorne (thorne@math.sc.edu)

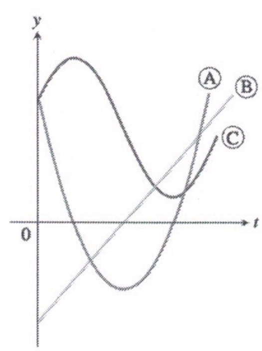
Monday, October 30, 2023

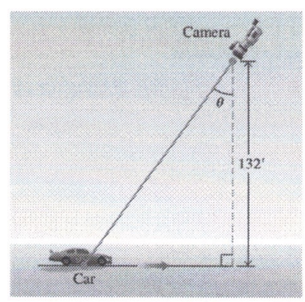
Please work without books, notes, calculators, phones, or assistance from others. If you have any questions, ask. Please do your work on separate paper and turn that in.

20 points each question.

GOOD LUCK!

(1) The graphs in the first figure below show the position s, velocity $s = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t. Which graph is which? Give reasons for your answers.





(2) You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mi/h (264 ft/sec), as shown in the second figure above. How fast will your camera angle θ be changing when the car is right in front of you?

(3) Use implicit differentiation to find dy/dx if $x^4 + \sin y = x^3y^2$.

(4) Sketch the graph of the inverse trigonometric function $y = \arctan(x)$. State its domain and range, and compute its derivative. When computing its derivative, explain your reasoning and please draw an appropriate triangle.

(5) Graph the function $y = f(x) = (\ln x)^2$. Answer the following questions as part of your solution.

(a) Where are the critical points of f?

(b) Where the local and absolute maxima and minima of f?

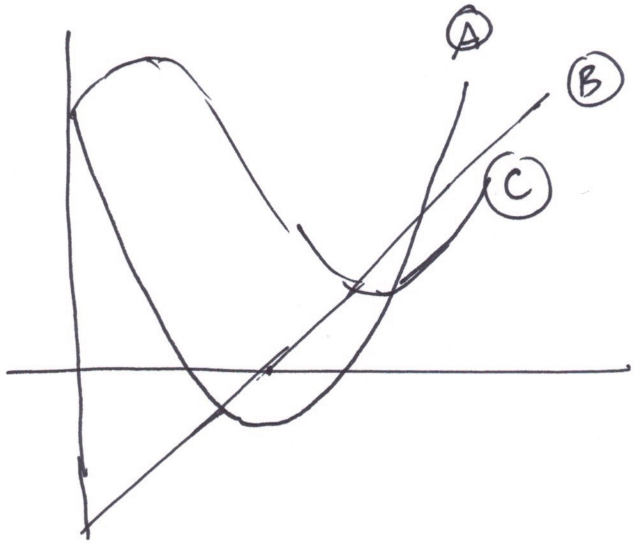
(c) Where are the inflection points of f?

(d) Where is f increasing and decreasing?

(e) Where is f concave up and down?

Midterm 2 - Math 141, Thorne, 10/30/23.

1.



C = position

B = acceleration

A = velocity

We can tell that (A) is the derivative of (c) because it has zeroes where (c) is flat (i.e. has critical points), and its sign matches where (C) is increasing + decreasing.

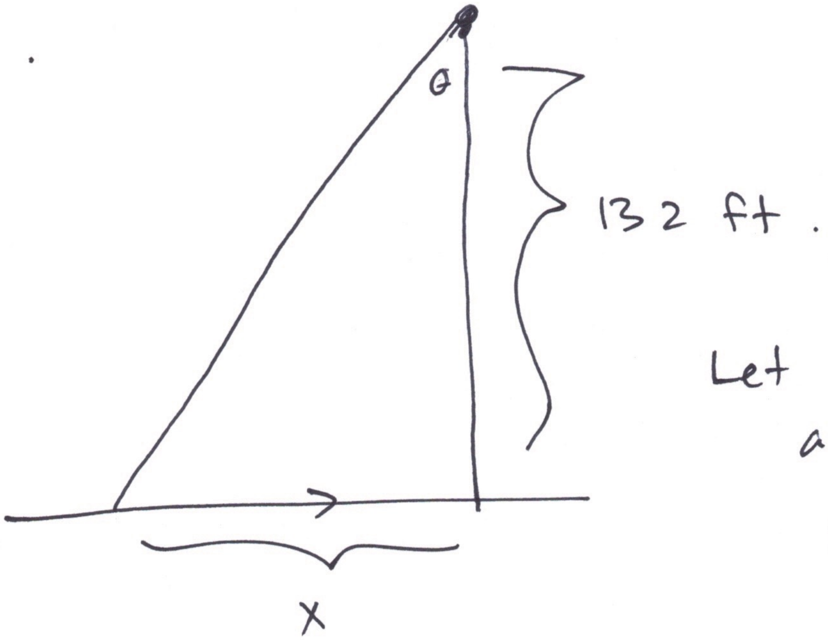
Similarly, (B) is the derivative of (A) as it has a single zero where the graph of (A) is flat, and its sign matches where (A) is increasing + decreasing.

The derivative of B would be constant, and is not graphed.

So, $\frac{d}{dt}(C) = (A)$, $\frac{d}{dt}(A) = (B)$

which means $(C) = s$, $(A) = \frac{ds}{dt}$ and $(B) = \frac{d^2s}{dt^2}$.

2.



Let θ = camera angle as in picture

x = distance along track from center

Know: $\frac{dx}{dt} = 264$ ft/s

Want: what is $\frac{d\theta}{dt}$?

We have $\tan(\theta) = \frac{x}{132}$

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{132}\frac{dx}{dt}$$

Right now $\begin{cases} \theta = 0 \text{ and } \sec(\theta) = 1 \\ \frac{dx}{dt} = 264 \end{cases}$

So get $\frac{d\theta}{dt} = 2$ radians/sec.

3. $x^4 + \sin y = x^3 y^2$

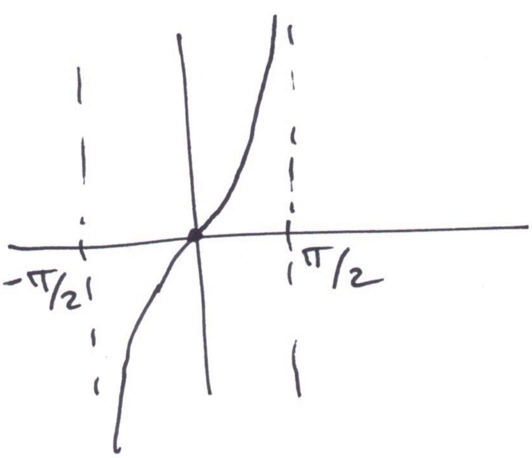
$$4x^3 + \cos y \frac{dy}{dx} = x^3 \cdot 2y \frac{dy}{dx} + 3x^2 \cdot y^2$$

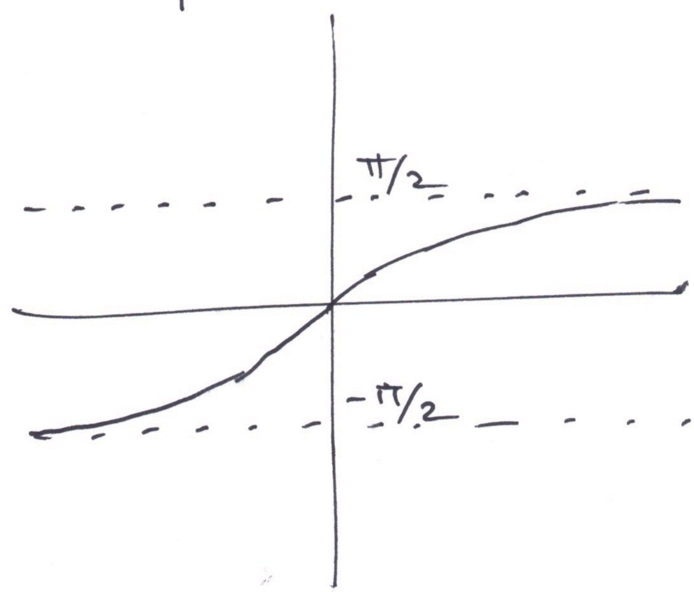
$$4x^3 - 3x^2 y^2 = x^3 \cdot 2y \frac{dy}{dx} \cancel{+} - \cos y \cdot \frac{dy}{dx}$$

$$= \frac{dy}{dx}\left(2x^3 y \cancel{+} - \cos y\right)$$

$$\frac{dy}{dx} = \frac{4x^3 - 3x^2 y^2}{2x^3 y + \cos y}.$$

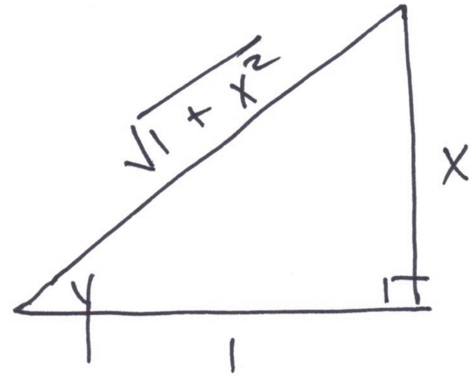
4. $y = \tan(x)$ (one period) $\qquad y = \arctan(x)$





Domain $(-\infty, \infty)$
Range $(-\pi/2, \pi/2)$.

To find derivative:



$$y = \arctan(x)$$

$$\tan(y) = x$$

$$\sec^2(y)\frac{dy}{dx} = 1$$

$$\cancel{\sec}\ \frac{dy}{dx} = \cos^2(y) = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}.$$

5. $y = (\ln x)^2$

$$\frac{dy}{dx} = \frac{2\ln x}{x}$$

$$\frac{d^2y}{dx^2} = \frac{x \cdot \frac{2}{x} - 2\ln x}{x^2} = \frac{2 - 2\ln(x)}{x^2}.$$

<u>Domain</u>: $(0, \infty)$, $\ln x$ not defined for $x \leq 0$.

Critical points: $\frac{dy}{dx}$ defined on its whole domain

Set $\frac{dy}{dx} = 0$: $\ln(x) = 0 \implies \boxed{x = 1.}$

Increasing/decreasing: For $x > 0$, the sign of $\frac{2\ln x}{x}$ is the same as that of $\ln(x)$: positive when $x > 1$, negative when $x < 1$.

Increasing on $(1, \infty)$
Decreasing on $(0, 1)$.

Set $\frac{d^2y}{dx^2} = 0$:

$$2 - 2\ln(x) = 0$$

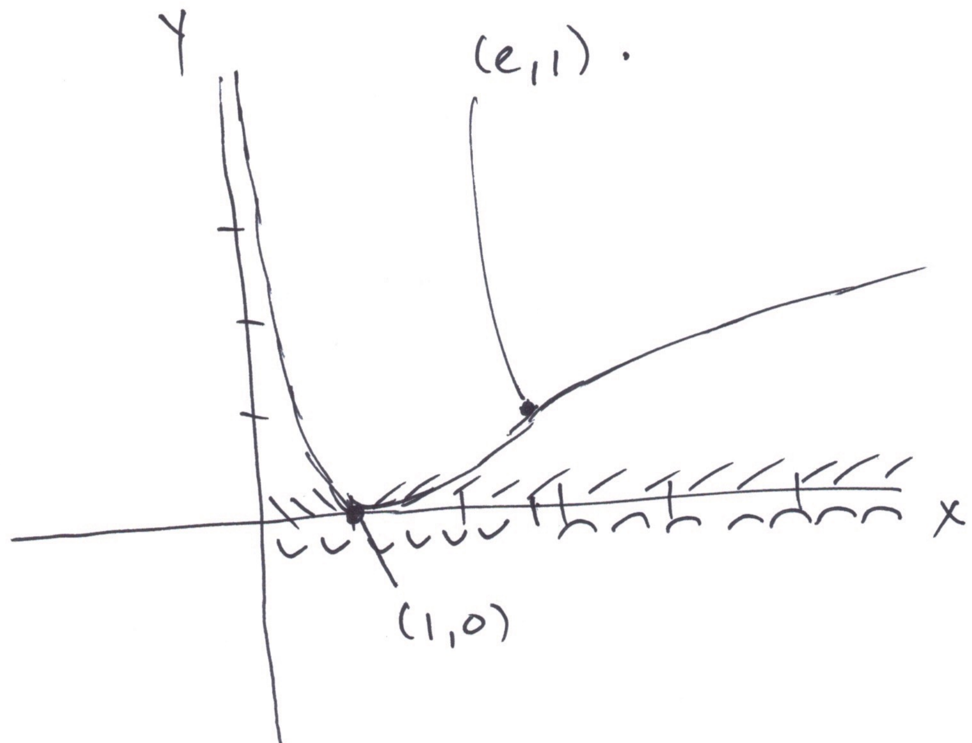
$$2 = 2\ln(x)$$

$$\ln(x) = 1, \text{ so } \boxed{x = e.}$$

For $x < e$, $\frac{2 - 2\ln(x)}{x^2} > 0$ so concave up

For $x > e$, $\frac{2 - 2\ln(x)}{x^2} < 0$ so concave down.

$x = e$ is a point of inflection.



$f(1) = (\ln 1)^2 = 0.$

$f(e) = (\ln e)^2 = 1.$

Local and absolute minimum at $x = 1$.

No local or absolute maxima.