Final Exam (Version 1) - Math 141, Frank Thorne (thorne@math.sc.edu)

Friday, December 11, 2023

Please work without books, notes, calculators, phones, or assistance from others. If you have any questions, ask. Please do your work on separate paper and turn that in.

GOOD LUCK!

(1) Give the definition of the derivative of a function $f(x)$ at the point $x = a$. (Please give the algebraic definition, using an equation.)

Draw a picture and explain why your equation gives the slope of the tangent line to the graph of $f(x)$ at $x = a$.

(2) What is the *definite integral* of a function $f(x)$, from $x = a$ to $x = b$? (Please give the algebraic definition, using an equation.)

Draw a picture and explain why your equation gives the signed area under the graph of $f(x)$ between $x = a$ and $x = b$.

(3) What does the Fundamental Theorem of Calculus say? (Both parts) Why is it important?

(4) A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

(5) Compute

$$\lim_{x\to\infty} (\ln x)^{1/x}.$$

(6) Evaluate

$$\int \frac{4+\sqrt{t}}{t^3} dt.$$

(7) Using rectangles, each of whose height is given by the value of the function at the midpoint of the rectangle's base (the midpoint rule), estimate the area under the graph of the function

$$f(x) = 1/x$$

from $x = 1$ to $x = 5$, using two rectangles.

Be sure to draw a picture and simplify your answer.

(8) Compute

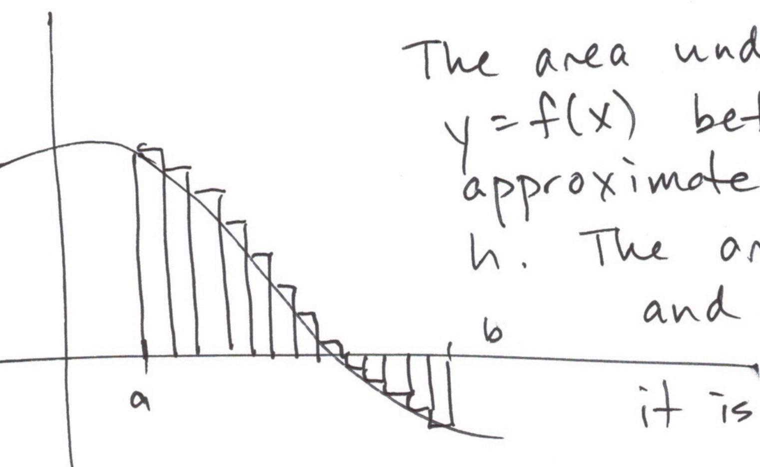
$$\int_0^{1/\sqrt{3}} \frac{dx}{1+4x^2}.$$

(9) Evaluate

$$\int x\sqrt{4-x}\, dx.$$

1. The definite integral $\int_a^b f(x)\,dx$, for a continuous function f, is defined by

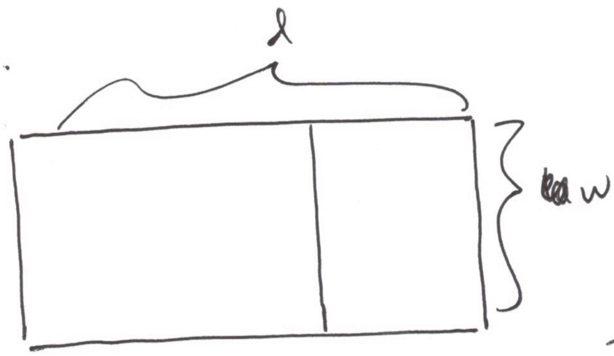
$$\int_a^b f(x)\,dx = \lim_{h\to 0}\Big(f(a)h + f(a+h)h + f(a+2h)h + \cdots + f(b-h)h\Big).$$



The area underneath the graph of $y=f(x)$ between $x=a$ and $x=b$ is approximated by rectangles of width h. The area of each is $f(x)\cdot h$, and this is signed because it is negative if $f(x)$ is negative.

Taking the total area of the rectangles gives the expression inside the limit above. As $h\to 0$, these rectangles better and better approximate the actual area so that the limit gives the true (signed) area.

2.



Set w = width of fence (with a parallel fence in the middle)

l = length of fence

Then $w\cdot l = 216\ m^2$.

The amount of fence is $F = 3w + 2l$.

As $wl = 216$, we can write $l = \frac{216}{w}$,

so $F = 3w + \frac{432}{w}$.

We have $\frac{dF}{dw} = 3 - \frac{432}{w^2}$

Set $= 0$: $3 = \frac{432}{w^2}$

$$w^2 = \frac{432}{3} = 144$$

$$w = 12 \text{ m and } \quad l = \frac{216}{12} = 18 \text{ m}.$$

The outer rectangle should be $\boxed{12 \text{ m} \times 18 \text{ m}}$

The total fence required is $3 \cdot 12 + 2 \cdot 18 = \boxed{72 \text{ m}}$

(3) Compute $\lim_{x \to \infty} (\ln x)^{1/x}$.

We have $\ln\left((\ln x)^{1/x}\right) = \frac{1}{x} \ln(\ln x)$.

By L'Hôpital's rule,

$$\lim_{x \to \infty} \frac{\ln(\ln x)}{x} = \lim_{x \to \infty} \frac{\frac{1}{\ln(x)} \cdot \frac{d}{dx}(\ln x)}{1}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{\ln(x)} \cdot \frac{1}{x}}{1}$$

$$= \lim_{x \to \infty} \frac{1}{\ln(x) \cdot x} = 0.$$

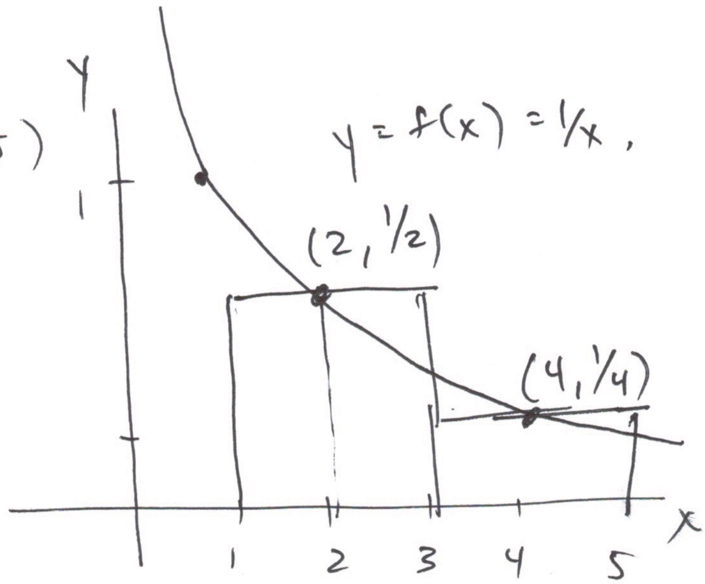
So, since we took a log, $\lim_{x \to \infty} (\ln x)^{1/x} = e^0 = 1$.

(4) $\displaystyle\int \frac{4+\sqrt{t}}{t^3}\,dt = \int \left(4t^{-3} + t^{-5/2}\right) dt$

$$= \frac{4t^{-2}}{-2} + \frac{t^{-3/2}}{-3/2} + C$$

$$= -2t^{-2} - \frac{2}{3}t^{-3/2} + C.$$

(5)



(not to scale)

The area of the rectangles is $\frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 2$

$$= 1 + \frac{1}{2} = \frac{3}{2},$$

so $\displaystyle\int_1^5 \frac{1}{x}\,dx \approx \frac{3}{2}$.

(6) We have $\displaystyle\frac{d}{dx}(\arctan(2x)) = \frac{2}{1+(2x)^2} = \frac{2}{1+4x^2}$

so $\displaystyle\int_0^{1/\sqrt{3}} \frac{dx}{1+4x^2} = \frac{1}{2}\arctan(2x)\Big|_0^{1/\sqrt{3}}$

$$= \frac{1}{2}\arctan\left(\frac{2}{\sqrt{3}}\right) - \frac{1}{2}\arctan(0)$$

$$= \frac{1}{2} \cdot \frac{\pi}{3} = \frac{\pi}{6}.$$

(7) $\int x\sqrt{4-x}\,dx$.

Set $u = 4-x$

$du = -dx$

$x = 4-u$

$$\int x\sqrt{4-x}\,dx = \int (4-u)\sqrt{u}\cdot(-du)$$

$$= \int (u-4)\sqrt{u}\,du$$

$$= \int \left(u^{3/2} - 4u^{1/2}\right) du$$

$$= \frac{2}{5}u^{5/2} - 4\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}(4-x)^{5/2} - \frac{8}{3}(4-x)^{3/2} + C.$$