Study Guide for AP Calc - Derivatives

Unit 2 – Limits and Continuity

- The limit of a function is the y value that the graph becomes closer to as the x value approaches a specific value, c
  - The graph must approach the same y value from both sides for the limit to exist
  - $\lim_{x \to c} f(x) = limit$
- Limit Definition of the Derivative
  - $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = mtan$
- Alternative Limit Definition of the Derivative
  - $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = mtan$
- Squeeze Theorem
  - If f(x) ≤ g(x) ≤ h(x) when x is near $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = limit$ then $\lim_{x \to a} g(x) = limit$
- Intermediate Value Theorem
  - If f is continuous on the interval [a,b], then every y value between is reached at some point
- Non – Differentiable Instances
  - Cusp
  - Sharp Corner
  - Vertical Tangent
  - Discontinuity

Unit 3 – Rules/Techniques of Differentiation

- Power Rule
  - If $f(x) = x^n$ where n is a rational number, then $f'(x) = nx^{n-1}$
    - $f(x) = x^3$ $f'(x) = 3x^2$
- Product Rule
  - f(x) * g(x) derivative is equal to f(x) * g'(x) + f'(x) * g(x)
    - $f(x) = (4x^2)(3x)$ $f'(x) = (4x^2)(3) + (8x)(3x)$ $f'(x) = 36x^2$
- Quotient Rule
  - $\frac{f(x)}{g(x)}$ derivative is equal to $\frac{g(x)*f'(x)-g'(x)*f(x)}{g(x)^2}$
    - $f(x) = \frac{3x+1}{x^2-1}$ $f'(x) = \frac{(x^2-1)(3)-(3x+1)(2x)}{(x^2-1)^2}$ $f'(x) = \frac{-3x^2-2x-3}{x^4-2x^2+1}$
- Chain Rule
  - Used to find derivative for a quantity raised to a power
  - $f(g(x)) = f'(g(x)) * g'(x)$
    - $f(x) = (2x+1)^3$ $f'(x) = 3(2x+1)^2 * (2)$ $f'(x) = 6(2x+1)^2$
- Trigonometric Derivatives

f(x) = sin(x)	f'(x) = cos(x)
f(x) = cos(x)	f'(x) = -sin(x)
f(x) = tan(x)	$f'(x) = \sec^2(x)$
f(x) = csc(x)	f'(x) = -csc(x) * cot(x)
f(x) = sec(x)	f'(x) = sec(x) * tan(x)
f(x) = cot(x)	$f'(x) = -\csc^2(x)$

- Inverse Trigonometric Derivatives

$f(x) = \sin^{-1}(x)$	$f'(x) = \frac{1}{\sqrt{1-x^2}}$
$f(x) = \cos^{-1}(x)$	$f'(x) = \frac{-1}{\sqrt{1-x^2}}$
$f(x) = \tan^{-1}(x)$	$f'(x) = \frac{1}{1+x^2}$
$f(x) = \csc^{-1}(x)$	$f'(x) = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
$f(x) = \sec^{-1}(x)$	$f'(x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$f(x) = \cot^{-1}(x)$	$f'(x) = \frac{-1}{1+x^2}$

- Logarithmic and Exponential Derivatives
  - $f(x) = a^x$ then the derivative would be $f'(x) = a^x \ln(a)$
    - $f(x) = 2^x$ $f'(x) = 2^x \ln(2)$
  - The derivative of natural base e^x is itself
    - $f(x) = e^x$ $f'(x) = e^x$
  - $y = \log_a x$ has a derivative of $y' = \frac{f'(x)}{f(x)*lna}$
    - $y = \log x^2$ $y' = \frac{2x}{x^2*ln10}$
  - The derivative of the natural log ln(x) is $y' = \frac{f'(x)}{f(x)}$
    - $y = \ln(x^3 + 1)$ $y' = \frac{3x^2}{x^3+1}$
- Derivative of $f^{-1}(x)$
  - Swap variables and solve for derivative
    - $y = \sqrt[3]{3x-5}$ $x = \sqrt[3]{3y-5}$ $y = \frac{x^3+5}{3}$ $y' = \frac{9x^2-x^3-5}{9}$
  - The slope of the derivative of f(x) is equal to the slope of the derivative of $f^{-1}(x)$ at partner points
- Differentiability vs. Continuity
  - Differentiable graphs must be continuous, but continuous graphs are not always differentiable
    - Not differentiable at vertical tangent, cusp, sharp corner, or discontinuity. Can still be continuous at cusp, corner, or vertical tangent.

Unit 4 – Concepts Involving the Derivative

- First Derivative Test to Determine Increasing/Decreasing intervals and Local Extrema
  - Find the derivative and its critical values (equal to zero or undefined), then place values on a number line
  - Choose a point in each interval and plug into first derivative equation to determine if it is positive or negative
    - Local extrema at critical points where slope changes sign
      - Positive to negative means local max
      - Negative to positive means local min
    - Increasing at Positive intervals and decreasing at negative intervals
- Second Derivative Test to Determine Concavity and Points of Inflection
  - Find second derivative and its critical values (equal to zero or undefined), then place on number line
  - Choose a point in each interval and plug into second derivative equation to determine if it is positive or negative
    - Points of inflection at critical points where sign changes
    - Concave up at positive intervals and concave down at negative intervals
- Summary of Connections from Graph of a Derivative
  - When graph is positive, the original is increasing, and when it is negative, the original is decreasing
  - Where graph crosses the x axis (y=0), there is a horizontal tangent
    - Crosses above to below means local max and crosses below to above means local min
  - Where graph is increasing the original is concave up, where graph is decreasing the original is concave down
  - Local extrema are points of inflection on original graph
- Mean Value Theorem
  - If f is continuous on a closed interval and differentiable on its interior, then there is at least one number where $\frac{f(b)-f(a)}{b-a} = f'(c)$
    - Find derivative of function
    - Find slope between a and b (msec)
    - Set equal and solve for x
- Extreme Value Theorem (Absolute Extrema)
  - Absolute extrema are max and min values of function in a specified interval
  - If f is continuous on a closed interval, then f attains both max and min value
    - Find derivative and critical values
    - Create table with critical values and plug into original equation
    - Identify absolute max and min based on y values
- Relationship Between Position, Velocity, and Acceleration
  - Position function s(t) calculates change as an average and change at an instant
  - Velocity function v(t) calculates change in position over time with direction
    - Positive if moving up or right
    - Negative if moving down or left
  - Acceleration function a(t) calculates change in velocity over time
- Total Distance vs. Displacement
  - Total distance is the total amount traveled
  - Displacement is the difference between where you start and where you end
- Speeding Up vs. Slowing Down
  - Speeding up at intervals where velocity and acceleration are both positive or both negative
  - Slowing down at intervals where velocity is positive, and acceleration is negative or vice versa

Unit 5 – Application of Derivatives

- L'Hopital's Rule
  - Suppose that f(a) = g(a) = 0, that f and g are differentiable on an open interval containing a, and that g'(x) ≠ 0 on the interval if x ≠ a. Then, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$
    - I
  - Indeterminate Forms (must result in one of these during direct substitution to apply rule)
    - $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, \infty - \infty, 0 * \infty, 0^0, 1^\infty, \infty^0$
  - $\lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0}$ $\lim_{x \to 0} \frac{\cos x}{1} = 1$
- Summary of Linear Approximation Method
  - Find equation of tangent line at a close x value near the value you would like to approximate
    - y-y1 = m (x-x1)
  - Place value into equation and solve
- Related Rates Summary
  - Situation where two or more variables that are closely related are changing with respect to time
    - Make a drawing and state the given
    - Determine rates of change given in context of the problem
    - Determine an equation that is appropriate for conditions of the problem
    - Find missing values for the moment in time being discussed
    - Find derivative of equation implicitly with respect to time
    - Substitute all known information and solve for desired rate of change
- Optimization Summary
  - Find values of controllable factors determining the behavior of a system that maximize productivity or minimize waste
    - State the given
    - Write a formula for maximized or minimized values
    - Rewrite formula in terms of a single variable
    - Find derivative and critical values
    - Find the maximum or minimum value