Examination 2 - Math 141, Frank Thorne (thornef@mailbox.sc.edu)

Friday, October 30, 2015, 10:50 a.m.

Please work without books, notes, calculators, or any assistance from others. If you have any questions, feel free to ask me. Please do your work on separate paper.

As always, simplify your answers, draw pictures where relevant, and put equals signs where they belong.

The first four questions are worth 15 points each, question 5 is worth 18 points, and question 6 is worth 22 points.

(1) Find $\frac{dy}{dx}$ if

$$y = \frac{1}{18}(3x-2)^6 + \left(4 - \frac{1}{2x^2}\right)^{-1}.$$

(2) Use implicit differentiation to find $\frac{dy}{dx}$, given that

$$xy = \cot(xy).$$

(3) Find $\frac{dy}{dx}$ if

$$y = \ln(\sin x).$$

(4) Find the extreme values (absolute and local) of the function and where they occur.

$$y = x^3 - 2x + 4$$

(5) A girl flies a kite at a height of 300 feet, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

Your answer must include a diagram and explanations of all notation you introduce.

(6) Graph the function

$$x^{2/3}(x-5).$$

Follow the seven-step procedure from your book, which is displayed on the overhead projector.

1. $\dfrac{dy}{dx} = \dfrac{1}{18} \cdot 6(3x-2)^5 \cdot \dfrac{d}{dx}(3x-2) + (-1)\left(4 - \dfrac{1}{2x^2}\right)^{-2}$

$\cdot \dfrac{d}{dx}\left(4 - \dfrac{1}{2x^2}\right)$

$= (3x-2)^5 - \left(4 - \dfrac{1}{2x^2}\right)^{-2} \cdot \dfrac{-1}{2} \cdot \dfrac{-2}{x^3}$

$= (3x-2)^5 - \dfrac{1}{x^3}\left(4 - \dfrac{1}{2x^2}\right)^{-2}$

//

2. $x\dfrac{dy}{dx} + y = -\csc^2(xy) \cdot \left[x\dfrac{dy}{dx} + y\right]$

$x\dfrac{dy}{dx} + y = -x\csc^2(xy)\dfrac{dy}{dx} - y\csc^2(xy)$

$x\dfrac{dy}{dx} + x\csc^2(xy)\dfrac{dy}{dx} = -y - y\csc^2(xy)$

$\dfrac{dy}{dx}\left(x + x\csc^2(xy)\right) = -y - y\csc^2(xy)$

$\dfrac{dy}{dx} = \dfrac{-y - y\csc^2(xy)}{x + x\csc^2(xy)} = \dfrac{-y(1+\csc^2(xy))}{x(1+\csc^2(xy))} = \dfrac{-y}{x}.$

10:50/

3. $\frac{dy}{dx} = \frac{1}{\sin(x)} \cdot \frac{d}{dx}(\sin x) = \frac{\cos x}{\sin x} = \cot x.$

4. $y = x^3 - 2x + 4$

$\frac{dy}{dx} = 3x^2 - 2.$ This is never undefined

no endpoints

zero if $3x^2 = 2$, $x^2 = \frac{2}{3}$

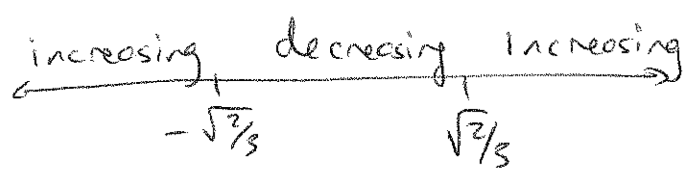
$x = \pm\frac{\sqrt{2}}{\sqrt{3}}.$

Critical points : $x = \pm\sqrt{\frac{2}{3}}.$

Choose sample points $x = -1$ $\quad \frac{dy}{dx} > 0 \quad (= +1)$

$x = 0 \quad \frac{dy}{dx} < 0 \quad (= -2)$

$x = 1 \quad \frac{dy}{dx} > 0 \quad (= 1)$

So 

Function has a local min at $x = \sqrt{\frac{2}{3}}$

$y = \left(\sqrt{\frac{2}{3}}\right)^3 - 2\cdot\sqrt{\frac{2}{3}} + 4$

$= \frac{2}{3}\sqrt{\frac{2}{3}} - 2\sqrt{\frac{2}{3}} + 4$

$= 4 - \frac{4}{3}\sqrt{\frac{2}{3}}$

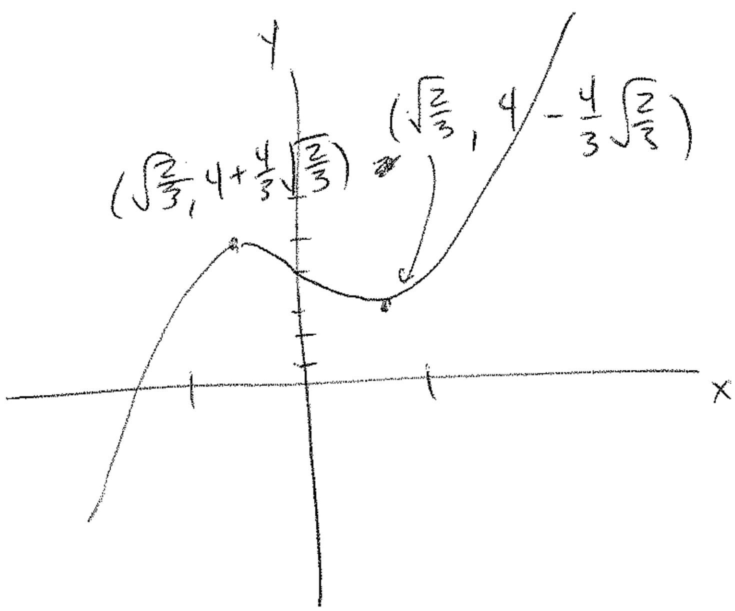
4 (cont.)

Function has a local max at $x = -\sqrt{\frac{2}{3}}$

$$y = \left(-\sqrt{\tfrac{2}{3}}\right)^3 - 2\cdot\left(-\sqrt{\tfrac{2}{3}}\right) + 4$$

$$= -\frac{2}{3}\sqrt{\frac{2}{3}} + 2\cdot\sqrt{\frac{2}{3}} + 4$$

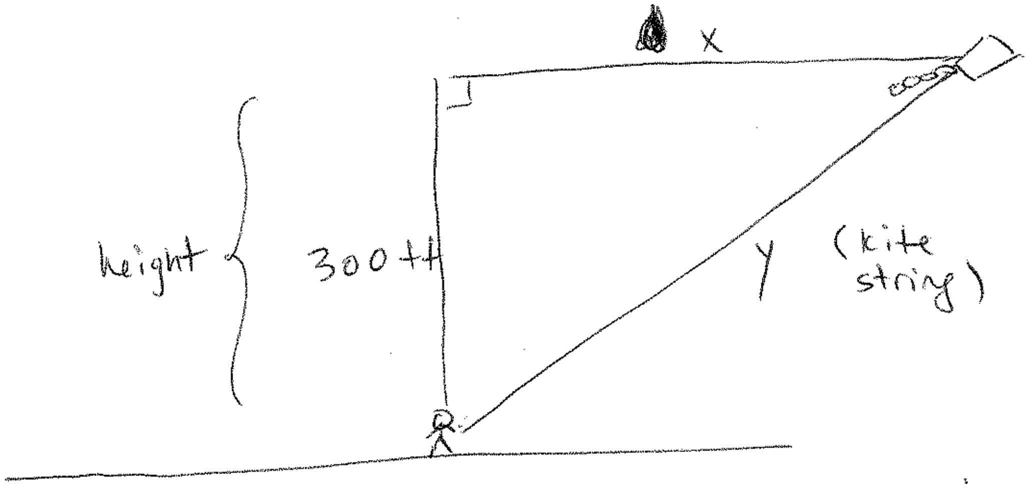
$$= 4 + \frac{4}{3}\sqrt{\frac{2}{3}}$$



No global extrema because $\lim_{x \to \infty} x^3 - 2x + 4 = \infty$

$\lim_{x \to -\infty} x^3 - 2x + 4 = -\infty$.

5.



Let y = length of kite string
Let x = horizontal distance from girl

$300^2 + x^2 = y^2$ by Pythagorean Thm

So $2x \frac{dx}{dt} = 2y \frac{dy}{dt}$

Right now $y = 500$ ft

$x = \sqrt{500^2 - 300^2} = 400$ ft

$\frac{dx}{dt} = 25$

So $2 \cdot 400 \cdot 25 = 2 \cdot 500 \frac{dy}{dt}$

$4 \cdot 25 = 5 \frac{dy}{dt}$ $\quad \frac{dy}{dt} = \frac{100}{5}$

$= \boxed{20 \text{ ft}/\text{s}.}$

Alternative interpretation: If you assumed x = 500 ft now, that is okay if you make your answer clear. (You get a different answer!)

6. Domain is all of $\mathbb{R}$, no obvious symmetries. (2 pts)

$$y = x^{2/3}(x-5) = x^{5/3} - 5x^{2/3}$$

$$\frac{dy}{dx} = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-2)$$ (2 pts)

$$\frac{d^2y}{dx^2} = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x+1)$$

Critical points where $\frac{dy}{dx} = 0$ (at $x=2$)
or undefined (at $x=0$)

Find sign of $\frac{dy}{dx}$:

	$x^{-1/3}$	$x-2$	$\frac{5}{3}x^{-1/3}(x-2)$
$(-\infty, 0)$	$-$	$-$	$+$
$(0, 2)$	$+$	$-$	$-$
$(2, \infty)$	$+$	$+$	$+$

(5 pts)

So increasing on $(-\infty, 0)$ and $(2, \infty)$
decreasing on $(0, 2)$.
Hence get a local min at 2
local max at 0

Possible pts of inflection where $\frac{dy}{dx} = 0$ (at $x=-1$)
or undefined (at $x=0$)

(5 pts)

Sign of $\frac{d^2y}{dx^2}$

	$x^{-4/3}$	$x+1$	$\frac{d^2y}{dx^2}$
$(-\infty, -1)$	$+$	$-$	$-$
$(-1, 0)$	$+$	$+$	$+$
$(0, \infty)$	$+$	$+$	$+$

Inflection point at $x=-1$
Concave up btw. ~~[illegible]~~ -1 and ∞, down btw. $-\infty$ and -1.

6 cont.

[No vertical asymptotes (function is defined everywhere)
(1 pt)
[No horiz asymptotes

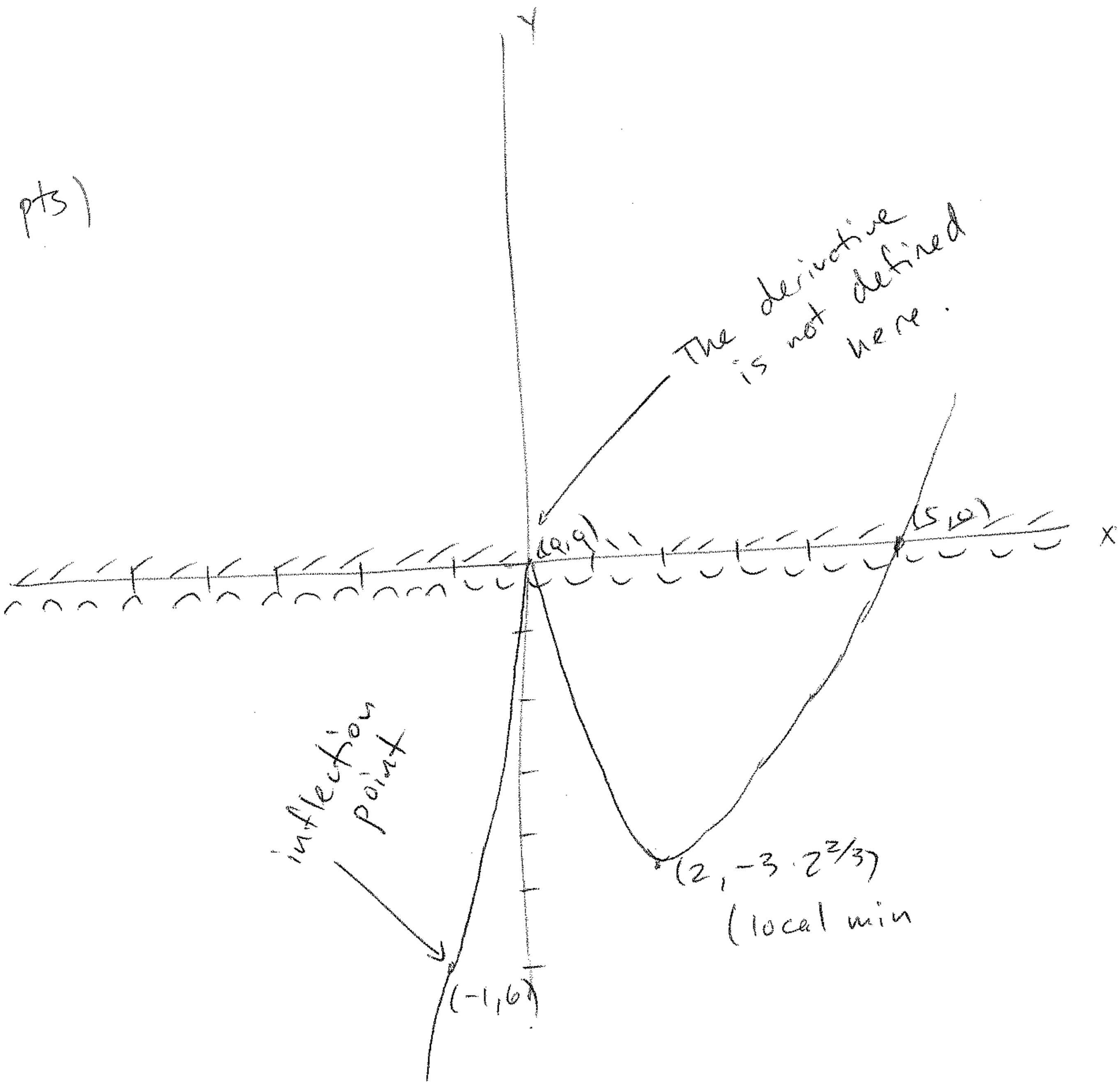
At $x=0$, $y=0^{5/3}-5\cdot 0^{2/3}=0$

At ~~x=~~
$x=-1$, $y=(-1)^{2/3}(-1-5)=-6$

At $x=2$, $y=2^{2/3}(2-5)=-3\cdot 2^{2/3}$

At $x=5$, $y=5^{2/3}(5-5)=0$.

(7 pts)



Examination 2 - Math 141, Frank Thorne (thornef@mailbox.sc.edu)

Friday, October 30, 2015, 12:00 p.m.

Please work without books, notes, calculators, or any assistance from others. If you have any questions, feel free to ask me. Please do your work on separate paper.

As always, simplify your answers, draw pictures where relevant, and put equals signs where they belong.

The first four questions are worth 15 points each, question 5 is worth 18 points, and question 6 is worth 22 points.

(1) Find $\frac{dy}{dx}$ if

$$y = (4x+3)^4(x+1)^{-3}.$$

(2) Use implicit differentiation to find $\frac{dy}{dx}$, given that

$$x + \tan(xy) = 0.$$

(3) Find $\frac{dy}{dx}$ if

$$y = (\ln x)^3.$$

(4) Find the extreme values (absolute and local) of the function and where they occur.

$$y = x^3 + x^2 - 8x + 5$$

(5) Sand falls from a conveyor belt at the rate of 10 $\mathrm{m}^3/\mathrm{min}$ onto the top of a conical pile. The height of the pile is always three eighths of the base diameter. How fast is the height changing when the pile is 4 m high?

Your answer must include a diagram and explanations of all notation you introduce. In the book, you were requested to answer in centimeters per minute. You may do so, or answer in meters per minute.

(6) Graph the function

$$x^{2/3}(x-5).$$

Follow the seven-step procedure from your book, which is displayed on the overhead projector.

1. $$\frac{dy}{dx} = (4x+3)^4 \frac{d}{dx}\left[(x+1)^{-3}\right] + \frac{d}{dx}\left[(4x+3)^4\right](x+1)^{-3}$$

$$\cancel{= 4 \cdot 4 \cdot (4x}$$

$$= (4x+3)^4 \cdot (-3)(x+1)^{-4} + 4 \cdot 4 \cdot (4x+3)^3 \cdot (x+1)^{-3}$$

$$= (x+1)^{-4}(4x+3)^3\left[(-3)(4x+3) + 16(x+1)\right]$$

$$= (x-1)^4(4x+3)^3\left[-12x - 9 + 16x + 16\right]$$

$$= (x-1)^4(4x+3)^3(4x+7).$$

2. $$1 + \sec^2(xy)\frac{d}{dx}(xy) = 0$$

$$1 + \sec^2(xy)\left[x\frac{dy}{dx} + y\right] = 0$$

$$1 + x\sec^2(xy)\frac{dy}{dx} + y\sec^2(xy) = 0$$

$$x\sec^2(xy)\frac{dy}{dx} = -y\sec^2(xy) - 1$$

$$\frac{dy}{dx} = \frac{-y\sec^2(xy) - 1}{x\sec^2(xy)}.$$

3. $$\frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{d}{dx}(\ln x) = \frac{3(\ln x)^2}{x}.$$

9. $\dfrac{dy}{dx} = 3x^2 + 2x - 8$.

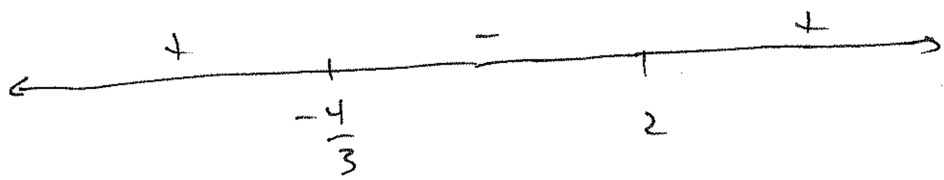
The zeroes are $x = \dfrac{-2 \pm \sqrt{4 - 4 \cdot 3 \cdot (-8)}}{6}$

$$= \frac{-2 \pm \sqrt{4 + 96}}{6}$$

$$= \frac{2 \pm \sqrt{100}}{6} = \frac{2 \pm 10}{6} = 2, \frac{-4}{3}.$$

i.e. $3x^2 + 2x - 8 = 3(x-2)\left(x + \frac{4}{3}\right)$

So the critical points are $x = 2$ and $x = \frac{-4}{3}$.



Choose sample points $x = -10$: $\dfrac{dy}{dx} = 300 - 20 - 8 > 0$

$x = 0$: $\dfrac{dy}{dx} = -8 < 0$

$x = 10$: $\dfrac{dy}{dx} = 300 + 20 - 8 > 0$.

So increasing on $(-\infty, \frac{-4}{3})$ and $(2, \infty)$

decreasing on $(\frac{-4}{3}, 2)$.

Hence the function has a local max at $x = \frac{-4}{3}$

$$y = \left(\frac{-4}{3}\right)^3 + \left(\frac{-4}{3}\right)^2 - 8 \cdot \left(\frac{-4}{3}\right) + 5$$

$$= \frac{-64}{27} + \frac{16}{9} + \frac{32}{3} + 5$$

$$= \frac{-64}{27} + \frac{48}{27} + \frac{288}{27} + \frac{135}{27} = \frac{439}{27}$$

a local min at $x = 2$

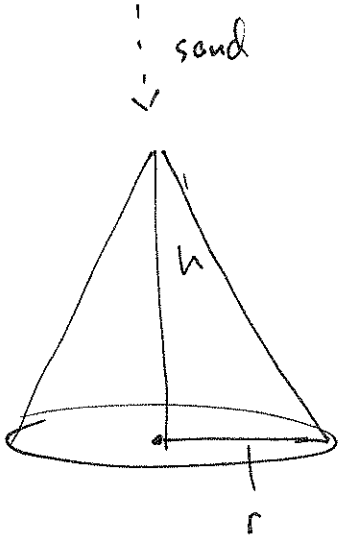
$$y = 2^3 + 2^2 - 8 \cdot 2 + 5$$
$$= 8 + 4 - 16 + 5 = 1.$$

No global extrema because

$$\lim_{x \to \infty} x^3 + x^2 - 8x + 5 = \infty$$

$$\lim_{x \to -\infty} x^3 + x^2 - 8x + 5 = -\infty.$$

5.



Let h = height of cone

r = radius

Then $h = \frac{3}{8} \cdot (2r) = \frac{3r}{4}$, $r = \frac{4h}{3}$.

Let $V = \frac{1}{3}\pi h r^2$ = volume of cone

and so $\cancel{V = \frac{1}{3}\pi\left(\frac{3r}{4}\right)r^2}$

$$V = \frac{1}{3}\pi h\left(\frac{4h}{3}\right)^2$$

$$= \frac{16\pi}{27} h^3.$$

So $$\frac{dV}{dt} = \frac{16\pi}{27} \cdot 3 \cdot h^2 \frac{dh}{dt}$$

$$= \frac{16\pi}{9} h^2 \frac{dh}{dt}.$$

Right now $\frac{dV}{dt} = 10$ m^3/min

$h = 4$ m

$$10 = \frac{16\pi}{9} \cdot 4^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = 10 \cdot \frac{9}{16\pi} \cdot \frac{1}{16}$$

$$= \frac{5 \cdot 9}{8 \cdot 16\pi} = \frac{45}{128\pi} \text{ m/min.}$$