Midterm Exam 1 - Math 142, Frank Thorne (thorne@math.sc.edu)

Thursday, September 19, 2019

Instructions and Advice:

- No books, notes, calculators, cell phones, or assistance from others.
- You are welcome to as much scratch paper as you need. Turn in everything you want graded. Whatever you don't want graded, put in a separate pile and I will recycle it.
- **Draw pictures where appropriate. Be clear, write neatly, explain what you are doing, and show your work.**
- 75 minutes is a long time. If you finish early, you have the opportunity to check your work.
- **Theorem.** If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error made in estimating $\int_a^b f(x)dx$ with the Trapezoidal Rule with n intervals is at most

$$\frac{M(b-a)^3}{12n^2}.$$

GOOD LUCK!

(1) What is the formula for integration by parts? Why does it work?

(2) Evaluate

$$\int_0^{\pi/2} \sin^2 x \, dx.$$

(3) Evaluate

$$\int \frac{5\, dx}{\sqrt{25x^2 - 9}}$$

for $x > 3/5$.

(4) Express the integrand as a sum of partial fractions and evaluate the integral:

$$\int \frac{dt}{t^3 - t^2 - 2t}.$$

(5) Evaluate

$$\int \frac{\ln x}{x^2} dx.$$

(6) Consider the integral

$$\int_1^4 \frac{dt}{t}.$$

(a) Use the Trapezoidal rule with $n = 3$ subintervals to estimate this integral. Draw a picture that represents the area you are computing.

(b) Determine a number of subintervals which would guarantee an error less than 10^{-4}, if you estimated this integral using the Trapezoidal Rule with that many subintervals.

1. The formula for integration by parts is

(HW 2, (b))

$$\int u\,dv = uv - \int v\,du.$$

It is true because it comes from the product rule.

We have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

So $$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

or $$uv = \int u\,dv - \int v\,du$$

which is the same thing rearranged.

2. $$\int_0^{\pi/2} \sin^2(x)\,dx = \int_0^{\pi/2} \frac{1-\cos(2x)}{2}\,dx$$

(8.3 #14)

$$= \left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]_0^{\pi/2}$$

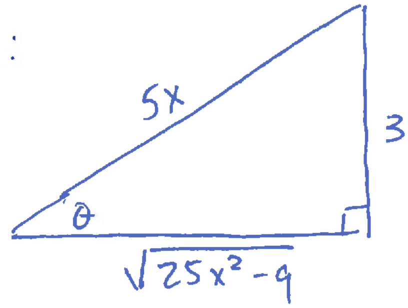
$$= \left(\frac{\pi/2}{2} - \frac{\sin(\pi)}{4}\right) - \left(\frac{0}{2} - \frac{\sin(0)}{4}\right)$$

$$= \left(\frac{\pi}{4} - 0\right) - (0 - 0) = \frac{\pi}{4}.$$

3. Evaluate $\int \frac{5\,dx}{\sqrt{25x^2-9}}$.

(8.4 #10)

Consider this triangle:



Then $\tan\theta = \frac{3}{\sqrt{25x^2-9}}$, so $\frac{5}{3}\tan\theta = \frac{5}{\sqrt{25x^2-9}}$

Also $\csc\theta = \frac{5x}{3}$, so $x = \frac{3}{5}\csc\theta$

$dx = -\frac{3}{5}\csc\theta\cot\theta\,d\theta$.

So $\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{5}{3}\tan\theta \cdot \left(-\frac{3}{5}\csc\theta\cot\theta\right) d\theta$

$= -\int \csc\theta\,d\theta$

$= \ln\left|\csc\theta + \cot\theta\right| + C$

$= \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$

4. $\displaystyle\int \frac{dt}{t^3-t^2-2t}$.
(8.5 #15)

We have $\displaystyle\frac{1}{t^3-t^2-2t} = \frac{1}{t(t^2-t-2)} = \frac{1}{t(t-2)(t+1)}$

$$= \frac{A}{t} + \frac{B}{t-2} + \frac{C}{t+1}$$

Get a common denominator:

$$\frac{A}{t} + \frac{B}{t-2} + \frac{C}{t+1} = \frac{A(t-2)(t+1) + Bt(t+1) + Ct(t-2)}{t(t-2)(t+1)}$$

$$= \frac{A(t^2-t-2) + B(t^2+t) + C(t^2-2t)}{t(t-2)(t+1)}$$

$$= \frac{t^2(A+B+C) + t(-A+B-2C) - 2A}{t(t-2)(t+1)}$$

Since this equals $\dfrac{1}{t^3-t^2-2t}$, we have $\begin{cases} A+B+C = 0 \\ -A+B-2C = 0 \\ -2A = 1. \end{cases}$

So $A = -1/2$.

Also, $\left.\begin{array}{l} B+C = \frac{1}{2} \\ B-2C = -1/2 \end{array}\right\} \Longrightarrow \begin{array}{rcl} 2B+2C &=& 1 \\ B-2C &=& -1/2 \\ \hline 3B &=& 1/2. \end{array}$

So $B = 1/6$

and $C = 1/2 - 1/6 = 1/3$.

Therefore

$$\int \frac{dt}{t^3-t^2-2t} = \int \left(\frac{-1/2}{t} + \frac{1/6}{t-2} + \frac{1/3}{t+1} \right) dt$$

$$= \frac{-1}{2} \ln|t| + \frac{1}{6} \ln|t-2| + \frac{1}{3} \ln|t+1| + C.$$

5. Evaluate $\displaystyle\int \frac{\ln x}{x^2}\, dx$.
(8.2 #35)

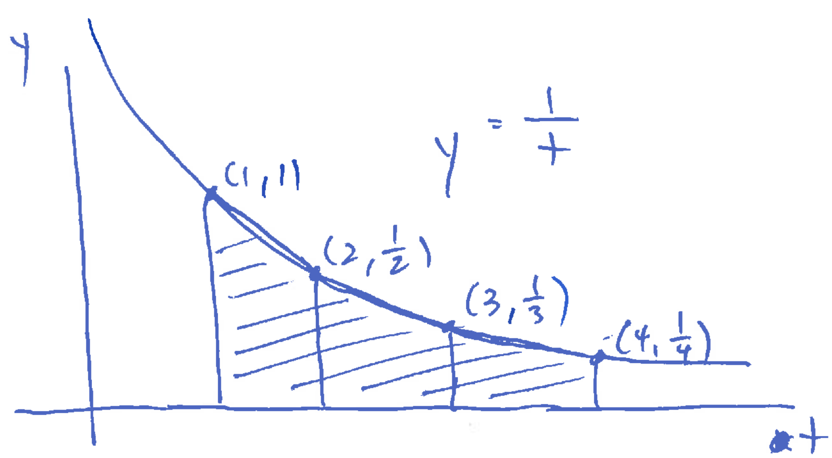
Set $u = \ln x$, $dv = \dfrac{dx}{x^2}$. So $du = \dfrac{1}{x}\, dx$, $v = \dfrac{-1}{x}$.

Then $\displaystyle\int \frac{\ln x}{x^2}\, dx = (\ln x)\left(\frac{-1}{x}\right) - \int \left(\frac{-1}{x}\right) \cdot \frac{1}{x}\, dx$

$$= -\frac{\ln x}{x} + \int \frac{1}{x^2}\, dx$$

$$= -\frac{\ln x}{x} - \frac{1}{x} + C.$$

6. $\displaystyle\int_1^4 \frac{dt}{t}$.



Trapezoidal rule gives area under the trapezoids, which is

$$1 \cdot \left(\frac{1}{2} \cdot 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{4} \right)$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{8} = \frac{12+12+8+3}{24} = \frac{35}{24}.$$

6(b) The error estimate is $\frac{M(b-a)^3}{12n^2}$.

Here $b=4$ and $a=1$.

M is an upper bound for $|f''|$ on $[1,4]$.

If $f(t)=\frac{1}{t}$, $f'(t)=\frac{-1}{t^2}$, $f''(t)=\frac{2}{t^3}$.

When $1 \le t \le 4$, t^3 is between 1 and 64.

So $\left|\frac{2}{t^3}\right|$ is bounded above by $\frac{2}{1}=2$.

The error estimate is $\frac{2\cdot(4-1)^3}{12\cdot n^2} = \frac{2\cdot 27}{12n^2} = \frac{27}{6n^2}$.

Want $\frac{27}{6n^2} < 10^{-4}$,

So $\frac{6n^2}{27} > 10^4$

So $n^2 > \frac{27}{6}\cdot 10^4$.

So $n > \sqrt{\frac{27}{6}\cdot 10^4} = 100\cdot\sqrt{\frac{27}{6}}$.

For example, $3^2 > \frac{27}{6}$, so $n > 300$ is good enough.