Midterm Exam 2 - Math 142, Frank Thorne (thorne@math.sc.edu)

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Instructions and Advice:

- No books, notes, calculators, cell phones, or assistance from others.
- You are welcome to as much scratch paper as you need. Turn in everything you want graded. Whatever you don't want graded, put in a separate pile and I will recycle it.
- **Draw pictures, and write complete sentences, where appropriate. Be clear, write neatly, explain what you are doing, and show your work. If (for example) you claim that a series converges or diverges, then thoroughly explain how you know.**
- Feel free to refer to the list of convergence tests provided with this exam.

GOOD LUCK!

HW 5, (b).

16 (1) Suppose f is continuous on $(-\infty, b]$. What does $\int_{-\infty}^{b} f(x)dx$ mean? What does it mean for it to converge or diverge?

17 (2) The integral (8.8, 32)

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}}$$

converges. Evaluate it.

17 (3) Does the series (10.3, #23)

$$\sum_{n=0}^{\infty} \frac{-2}{n+1}$$

converge or diverge? Explain how you know.

17 (4) Does the series (10.4, #29)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\ln n}$$

converge or diverge? Explain how you know.

16 (5) Does the series (similar to 10.5, 20)

$$\sum_{n=1}^{\infty} \frac{10^n}{n!}$$

converge or diverge? Explain how you know.

17 (6) Does the series (10.6, 18)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$$

converge absolutely, converge conditionally, or diverge? Explain how you know.

1. (HW #5, (b))

By definition, $\int_{-\infty}^{b} f(x)\,dx$ means $\lim\limits_{a \to -\infty} \int_a^b f(x)\,dx$.

The integral converges if this limit exists and diverges if the limit does not exist.

2. (8.8 #32) $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

The integrand is discontinuous at $x=1$ so the integral is improper:

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{|x-1|}} + \int_1^2 \frac{dx}{\sqrt{|x-1|}}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}}$$

$$= \lim_{a \to 1^-} \int_0^a \frac{dx}{\sqrt{1-x}} + \lim_{b \to 1^+} \int_b^1 \frac{dx}{\sqrt{x-1}}.$$

with $u = 1-x$ and $du = -dx$,

$$\int \frac{dx}{\sqrt{1-x}} = \int \frac{-du}{\sqrt{u}} = -2\sqrt{u} + C = -2\sqrt{1-x} + C$$

So $$\int_0^a \frac{dx}{\sqrt{1-x}} = -2\sqrt{1-x}\,\Big|_0^a = -2\sqrt{1-a} - (-2\sqrt{1-0})$$
$$= -2\sqrt{1-a} + 2$$

and $$\lim_{a \to 1^-} \int_0^a \frac{dx}{\sqrt{1-x}} = \lim_{a \to 1^-} \left(-2\sqrt{1-a} + 2\right)$$
$$= -2\sqrt{1-1} + 2 = 2.$$

Similarly,

$$\int_b^2 \frac{dx}{\sqrt{x-1}} = 2\sqrt{x-1}\Big|_b^2 = 2\sqrt{2-1} - 2\sqrt{b-1}$$
$$= 2 - 2\sqrt{b-1}$$

and $\lim_{b\to 1^+} \int_b^1 \frac{dx}{\sqrt{x-1}} = \lim_{b\to 1^+} (2 - 2\sqrt{b-1})$

$$= 2 - 2\sqrt{1-1} = 2.$$

So $\int_0^2 \frac{dx}{\sqrt{|x-1|}} = 2 + 2 = 4.$

3. It diverges.

The series converges if and only if $\sum_{n=0}^{\infty} \frac{2}{n+1}$ does.

Three ways to see that $\sum_{n=0}^{\infty} \frac{2}{n+1}$ diverges:

(1) Integral test. Look at $\int_0^{\infty} \frac{2}{t+1}\,dt$.

This is $\lim_{b\to\infty} \int_0^b \frac{2}{t+1}\,dt = \lim_{b\to\infty} 2\ln|t+1|\Big|_0^b$

$$= \lim_{b\to\infty} 2\ln|b+1| - 0$$
$$= \infty, \text{ i.e. it diverges}$$

So the sum $\sum_{n=0}^{\infty} \frac{2}{n+1}$ diverges.

(2) Comparison test. Compare to $\sum_{n=1}^{\infty} \frac{1}{n}$, ignoring the $n=0$ term.

We have $\frac{2}{n+1} \overset{?}{\geq} \frac{1}{n}$

$2n \overset{?}{\geq} n+1$

$n \geq 1$ ✓

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is smaller and diverges by the p-series test, $\sum_{n=0}^{\infty} \frac{2}{n+1}$ diverges also.

4. Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\ln(n)}$ converge or diverge?

This diverges. For example, we can use limit comparison.

We have
$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\ln(n)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n}\ln(n)}$$
$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\ln(n)}$$
$$= \lim_{n\to\infty} \frac{(1/2)\, n^{-1/2}}{1/n}$$
(by L'Hôpital)
$$= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{n}{n^{1/2}}$$
$$= \lim_{n\to\infty} \frac{1}{2}\sqrt{n} = \infty.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series (or Integral) test, our series diverges by the limit comparison test.

Bonus points / alternative answer: in fact, the question is not well defined because $\frac{1}{\sqrt{1}\ln(1)}$ does not exist ($\ln(1) = 0$). So, the question should have looked at $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln(n)}$. Strictly speaking it does not make sense.

5. Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{10^{n+1}}{10^n}\cdot\frac{n!}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|10\cdot\frac{1}{n+1}\right|$$

$$= 0.$$

So the series converges.

6. This is an alternating series. We have:

(1) The terms $\frac{1}{1+\sqrt{n}}$ are all positive,

(2) $\frac{1}{1+\sqrt{i}} \geq \frac{1}{1+\sqrt{i+1}}$ for all i, and

(3) $\lim_{n\to\infty}\frac{1}{1+\sqrt{n}} = 0$.

So by the alternating series test, it converges.

We have that $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{1+\sqrt{n}}\right| = \sum_{n=1}\frac{1}{1+\sqrt{n}}$ diverges.

Use the limit comparison test to compare to $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which diverges by the p-series test:

$$\lim_{n\to\infty} \frac{\frac{1}{1+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+1}$$

$$= \frac{1}{0+1} = 1.$$

So the series converges <u>conditionally.</u>