Midterm Exam 3 - Math 142, Frank Thorne (thorne@math.sc.edu)

Thursday, November 21, 2019

Instructions and Advice:

- There are six questions (**including on the back**).
- No books, notes, calculators, cell phones, or assistance from others.
- You are welcome to as much scratch paper as you need. Turn in everything you want graded. Whatever you don't want graded, put in a separate pile and I will recycle it.
- **Draw pictures, and write complete sentences, where appropriate. Be clear, write neatly, explain what you are doing, and show your work. If (for example) you claim that a series converges or diverges, then thoroughly explain how you know.**
- You are welcome to use any formulas for power series which you know. State explicitly any formula which you are using.
- If asked to compute a power series, either write in sigma notation or compute through (at least) the x^3 term.
- Feel free to refer to the list of convergence tests provided with this exam.

GOOD LUCK!

(1) What is a parametric curve? Give an example.

(2) Find the radius and interval of convergence for the series

$$\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}.$$

For what values of x does the series converge (a) absolutely, and (b) conditionally?

(3) Find the Maclaurin series (i.e., the Taylor series at $x = 0$) for the function

$$\frac{2+x}{1-x}.$$

(4) Find the Maclaurin series (i.e., the Taylor series at $x = 0$) for the function

$$xe^x.$$

(5) The attached sheet graphs the following four parametric curves:

(1) $x(t) = -2\cos(t),\ y(t) = 5\sin(t)$.

(2) $x(t) = 5\cos(t),\ y(t) = 2\sin(t)$.

(3) $x(t) = t,\ y(t) = 1 - t^2$.

(4) $x(t) = t - 1,\ y(t) = -t^2$.

(Note that the graphs are not on the same scale as each other.)

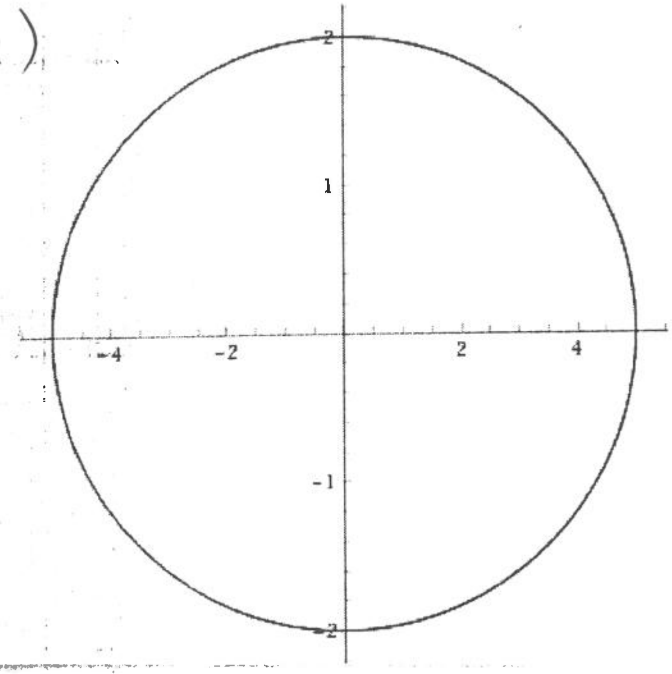
Match the equations to the graphs. Explain your reasoning.

(6) Find an equation for the line tangent to the curve

$$x = 4\sin(t),\ y = 2\cos(t)$$

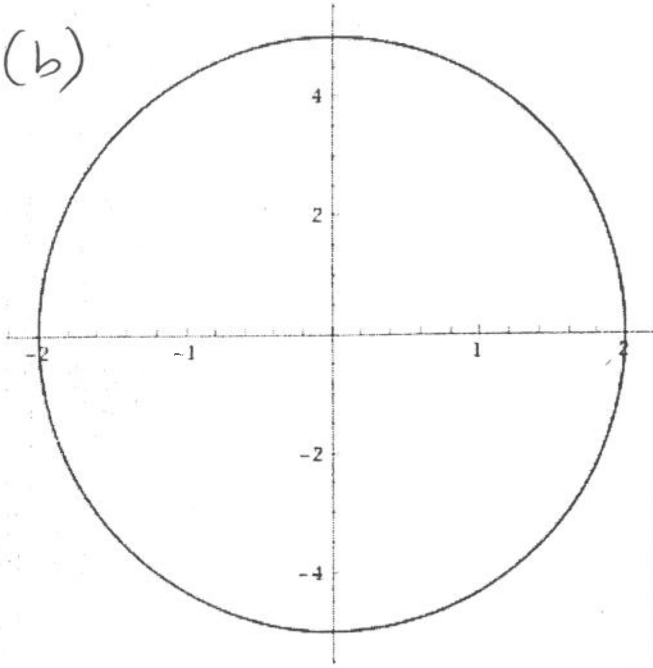
at the point $t = \pi/4$.

(a)



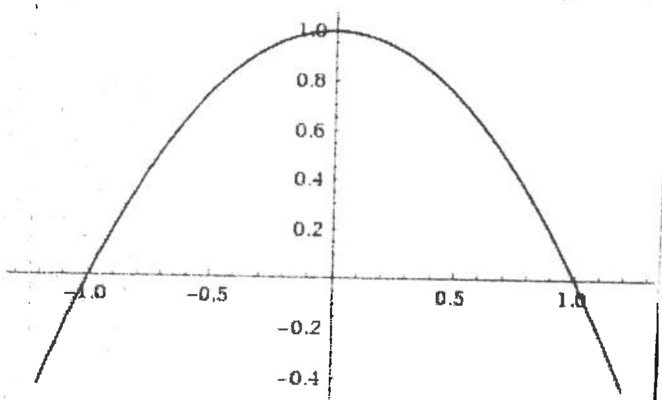
2
1
-1
-2
-4
-2
2
4

(b)



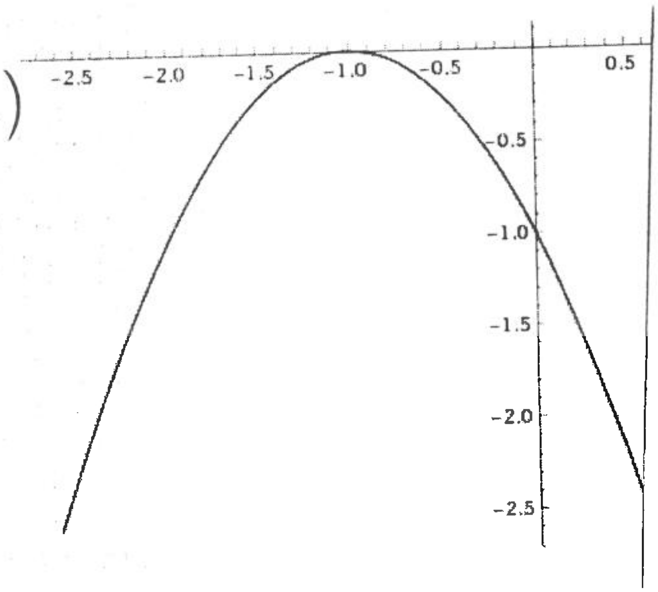
4
2
-2
-4
-2
-1
1
2

(c)



1.0
0.8
0.6
0.4
0.2
-0.2
-0.4
-1.0
-0.5
0.5
1.0

(d)



-2.5
-2.0
-1.5
-1.0
-0.5
0.5
-0.5
-1.0
-1.5
-2.0
-2.5

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1. A parametric curve is one where both x and y are given as functions of a third variable.
For example,

$$x = \cos(t), \quad y = \sin(t)$$

is a parametric curve — indeed, a circle.

2. $$\sum_{n=0}^{\infty} \frac{\sqrt{n}\, x^n}{3^n}$$

Apply the ratio test.

Let $$\rho := \lim_{n\to\infty} \left| \frac{\frac{\sqrt{n+1}\, x^{n+1}}{3^{n+1}}}{\frac{\sqrt{n}\, x^n}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{3^n}{3^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \sqrt{\frac{n+1}{n}} \cdot x \cdot \frac{1}{3} \right|$$

$$= \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} \cdot |x| \cdot \frac{1}{3}$$

$$= \frac{|x|}{3}.$$

It converges absolutely when $|\rho| < 1$, so when $\frac{|x|}{3} < 1$, so $|x| < 3$.

It diverges when $|\rho| > 1$, so when $|x| > 3$.

When $x = 3$, the series is $\sum_{n=0}^{\infty} \sqrt{n}$

When $x = -3$, the series is $\sum_{n=0}^{\infty} (-1)^n \sqrt{n}$.

In both cases the series diverges by the nth term test.

So: radius of convergence $= 3$

interval of convergence $= (-3, 3)$

Converges absolutely on $(-3, 3)$

Converges conditionally nowhere.

3. Maclaurin series for $\frac{2+x}{1-x}$

The Maclaurin series for $\frac{1}{1-x}$ is $1 + x + x^2 + x^3 + \cdots$

So, we have $\frac{2}{1-x} = 2 + 2x + 2x^2 + 2x^3 + \cdots$

$$\frac{x}{1-x} = x + x^2 + x^3 + x^4 + \cdots$$

and so by adding we see that

$$\frac{2+x}{1-x} = 2 + 3x + 3x^2 + 3x^3 + \cdots$$

4. We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

and so

$$xe^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} + \cdots$$

5.

(1) — (b). If $t = 0$, $x = -2$ and $y = 0$ and (b) is the only graph with $(-2, 0)$.

(2) — (a). If $t = 0$, $x = 5$ and $y = 0$ and (a) is the only graph with $(5, 0)$.

(3) — (c). If $t = 0$, $x = 0$ and $y = 1$, and (c) is the only graph with $(0, 1)$.

(4) — (d) by process of elimination. ⌣̈

Alternatively, $y(t) = -t^2$ is never positive, and (d) is the only graph that doesn't go above the x-axis.

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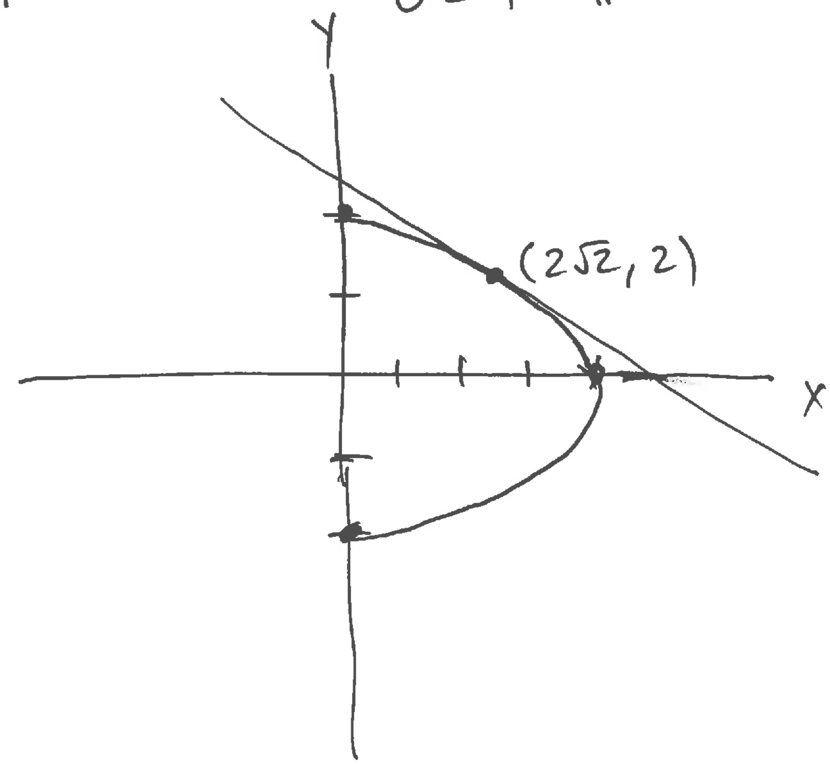
6. We draw a graph first.

t	x	y
0	0	2
$\pi/4$	$2\sqrt{2}$	$\sqrt{2}$
$\pi/2$	4	0
$3\pi/4$	$2\sqrt{2}$	$-\sqrt{2}$
π	0	-2

$0 \le t \le \pi$



If we graphed from $t = \pi$ to $t = 2\pi$, we would also get the left side of the ellipse.

The slope of the tangent line is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin(t)}{4\cos(t)} = \frac{-1}{2}\tan(t)$$

If we plug in $t = \pi/4$ we get

$$\frac{dy}{dx} = \frac{-1}{2}\tan\left(\frac{\pi}{4}\right) = \frac{-1}{2}\cdot 1 = \frac{-1}{2}.$$

So the tangent line is

$$y - \sqrt{2} = \frac{-1}{2}(x - 2\sqrt{2})$$

$$y - \sqrt{2} = \frac{-1}{2}x + \sqrt{2}$$

$$\boxed{y = \frac{-1}{2}x + 2\sqrt{2}}.$$