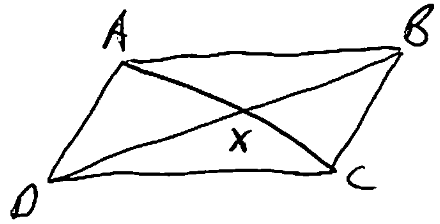
Solutions 1D

Problems 4, 6, 7, 8, 9, 10; challenge probs: 12, 13.



(4) Let $ABCD$ be a parallelogram. Suppose that the diagonals are perpendicular. Show: $ABCD$ is a rhombus.

Proof: We suppose that $ABCD$ is a parallelogram and that diagonals $AC \perp BD$. To show that $ABCD$ is a rhombus, it suffices to show that the 4 sides are equal.

We have:

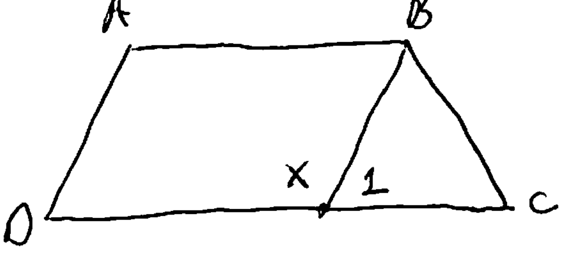
- $ABCD$: parallelogram $\Longrightarrow DX = XB$ (diags. bisect each other)
- $AC \perp BD \Longrightarrow \angle AXD = \angle AXB = 90^\circ$.

We therefore consider:

$$\left.\begin{array}{ccc} \underline{\triangle AXD} & & \underline{\triangle AXB} \\ AX & = & AX \\ \angle AXD & = & \angle AXB \\ DX & = & XB \end{array}\right\} \begin{array}{l} \xrightarrow{\text{SAS}} \triangle AXD \cong \triangle AXB \\ \xrightarrow[\text{sides}]{\text{corr.}} AD = AB. \end{array}$$

We now have:

$$\left.\begin{array}{l} ABCD \text{ parallelogr.} \Longrightarrow \left\{\begin{array}{l} AB = CD \\ AD = BC \end{array}\right\} \text{opp. sides} \\ AD = AB \end{array}\right\} \Longrightarrow \begin{array}{l} \text{all 4 sides equal;} \\ ABCD \text{ is a rhombus.} \end{array}$$

(6) 

Suppose that $AB \parallel CD$, but that $AD \nparallel BC$.

Show: $\angle D = \angle C \iff AD = BC$.

<u>Proof</u>: Draw X on DC with $AD \parallel BX$. Then we have $ABXD$ is a parallelogram (opp. sides parallel).

It follows that $\left.\begin{array}{l} AB = DX \\ AD = BX \end{array}\right\}$ (opp. sides equal).

($\Longrightarrow$) Suppose: $\angle D = \angle C$. To show: $AD = BC$.

To start, $AD \parallel BX \Longrightarrow \angle 1 = \angle D$ (corr. angles).

We have: $\left.\begin{array}{l} \angle D = \angle C \\ \angle 1 = \angle D \end{array}\right\} \Longrightarrow \angle 1 = \angle C \Longrightarrow \Delta BXC$ is isosc., base CX

$\Longrightarrow BX = BC$. Observe: $\left.\begin{array}{l} AD = BX \\ BX = BC \end{array}\right\} \Longrightarrow AD = BC$.

($\Longleftarrow$) Suppose: $AD = BC$. To show: $\angle D = \angle C$.

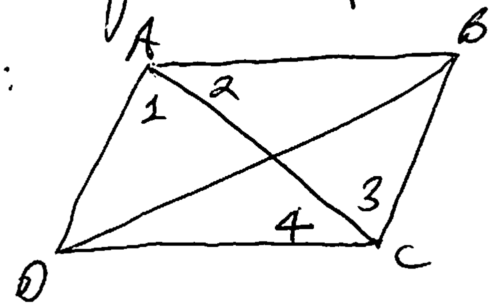
We have: $\left.\begin{array}{l} AD = BC \\ AD = BX \end{array}\right\} \Longrightarrow BC = BX \Longrightarrow \Delta BXC$ is isosc., base CX

$\xrightarrow[\text{asinorum}]{\text{pons}} \angle C = \angle 1$. But also note that $\angle D = \angle 1$ since $AD \parallel BX$ (corr. angles)

It follows that $\left.\begin{array}{l} \angle D = \angle 1 \\ \angle C = \angle 1 \end{array}\right\} \Longrightarrow \angle C = \angle D$.

(7) Let $ABCD$ be a parallelogram. Show that opposite interior angles are equal.

Proof:



It suffices to show that

- $\angle A = \angle C$
- $\angle B = \angle D$.

Draw diagonals AC and BD.

$$ABCD \text{ is a parallelogram} \underset{\text{def.}}{\Longrightarrow} \begin{cases} AD \parallel BC \Longrightarrow \angle 1 = \angle 3 \\ AB \parallel CD \Longrightarrow \angle 2 = \angle 4 \end{cases} \text{(alt. int. angles)}$$

We now have $\angle A = \angle 1 + \angle 2 = \angle 3 + \angle 4 = \angle C$. I.e., we have $\angle A = \angle C$.

Similarly, one can show that $\angle B = \angle D$.

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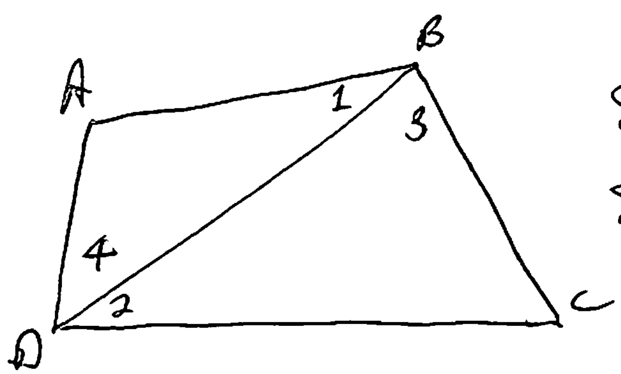
(8) Let $ABCD$ be a quadrilateral. Suppose that $AB \parallel CD$ and $\angle B = \angle D$. Show: $ABCD$ is a parallelogram.

Proof:



Since $AB \parallel CD$, it suffices to show that $AD \parallel BC$.

(then opp. sides are parallel).

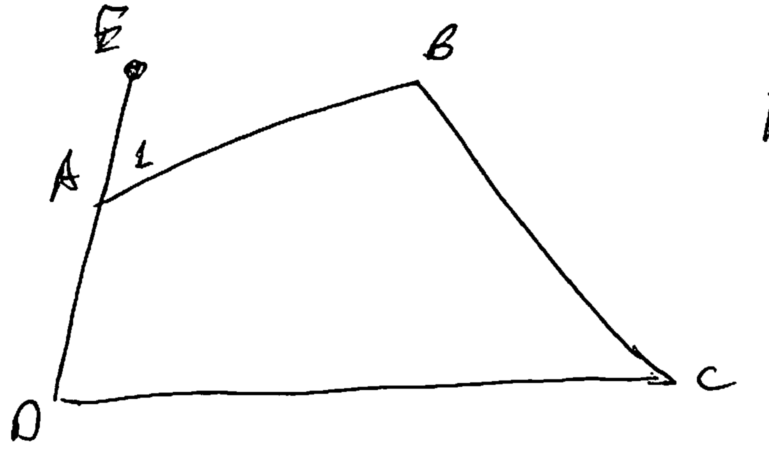
We have: $AB \parallel CD \Longrightarrow \angle 1 = \angle 2$ (alt. int. angles)

$$\left.\begin{aligned} \text{It follows that } \angle B &= \angle 1 + \angle 3 = \angle 2 + \angle 4 = \angle D \\ \text{and } \quad \angle 1 &= \angle 2 \end{aligned}\right\} \overset{\text{subtract}}{\Longrightarrow} \angle 3 = \angle 4$$

$\Longrightarrow AD \parallel BC$ (alt. int. angles)

(9) Let ABCD be a quadrilateral. Suppose that $\angle A = \angle C$ and $\angle B = \angle D$. Show: ABCD is a parallelogram.

Proof:



We will show: $AB \parallel CD$ and $AD \parallel BC$.

Sum of int. angles: $\angle A + \angle B + \angle C + \angle D \underset{\substack{\uparrow \\ \angle A = \angle C,\ \angle B = \angle D}}{=} 2\angle A + 2\angle B = 360^\circ$

$\overset{\div 2}{\Longrightarrow} \angle A + \angle B = 180^\circ$.

We now have: $\left.\begin{array}{l} \angle A + \angle B = 180^\circ \\ \angle A + \angle 1 = 180^\circ \end{array}\right\} \overset{\text{subtract}}{\Longrightarrow} \angle B - \angle 1 = 0^\circ$

$\Longrightarrow \angle B = \angle 1$

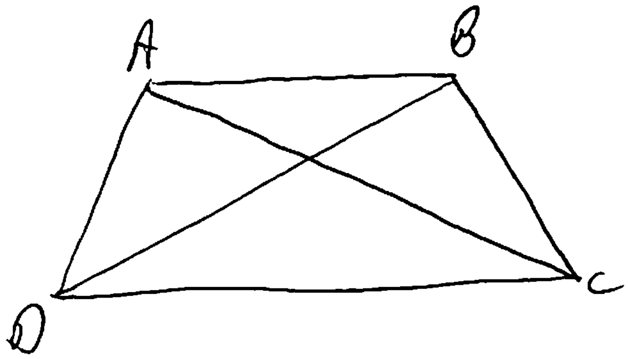
$\Longrightarrow ED = AD \parallel BC$ (alt. int. angles)

Similarly, one can show that $AB \parallel CD$.

$\left.\begin{array}{l} AB \parallel CD \\ AD \parallel BC \end{array}\right\} \Longrightarrow$ ABCD is a parallelogram.

(10) Let $ABCD$ be an isosceles trapezoid. Show: its diagonals are equal.

Proof:



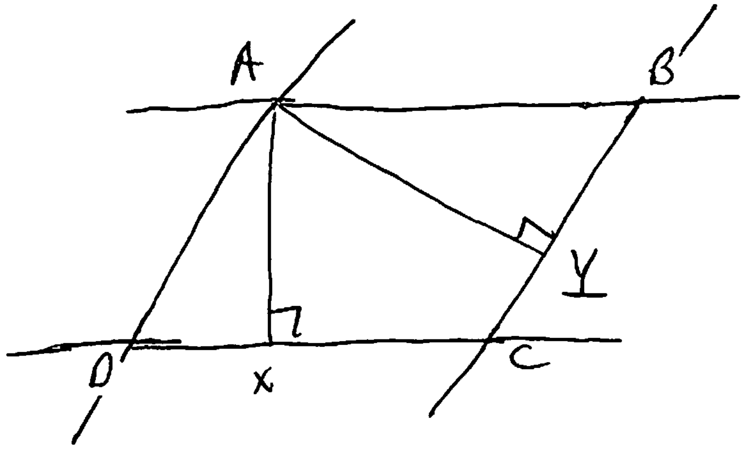
Suppose: $AB \parallel CD$, $AD = BC$.

To show: $AC = BD$.

10.6 : $AD = BC \Longrightarrow \angle D = \angle C$. Consider:

$$\left.\begin{array}{ccc} \underline{\triangle ADC} & & \underline{\triangle BCD} \\ AD & = & BC \\ \angle D & = & \angle C \\ DC & = & DC \end{array}\right\} \begin{array}{l} \xRightarrow{\text{SAS}} \triangle ADC \cong \triangle BCD \\ \xRightarrow[\text{sides}]{\text{corr.}} AC = BD. \end{array}$$

(13)



Suppose: ABCD is a parallelogram, and $AX = AY$.

Show: ABCD is a rhombus.

Proof: Since ABCD is a parallelogram,

$\left.\begin{array}{l} AB = CD \\ AD = BC \end{array}\right\}$ (opp. sides equal) $\Longrightarrow$ suffices to show that $AD = AB$. (then all 4 sides will be equal).

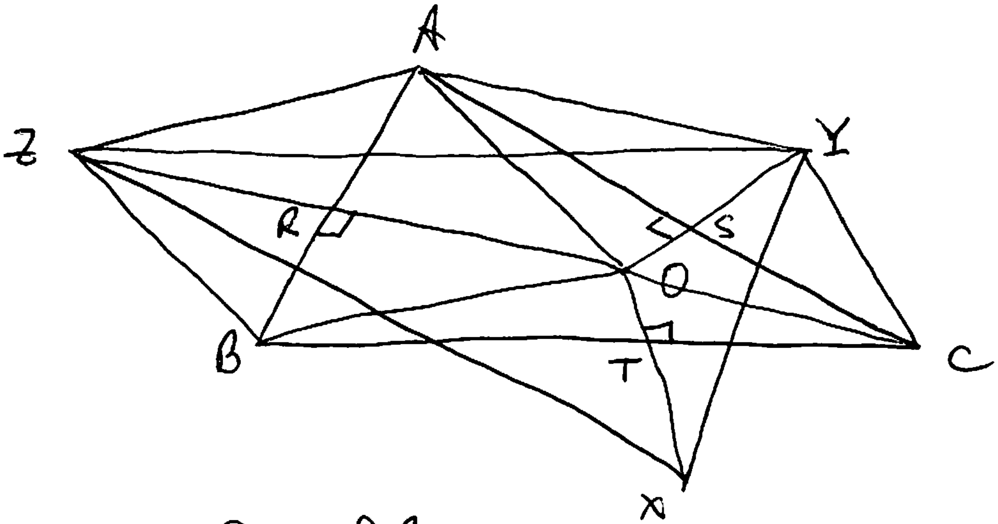
$\angle B = \angle D$ (opp. angles equal by LD.9)

Consider: $\underline{\Delta AXD}$ $\underline{\Delta AYB}$

$$\left.\begin{array}{lcl} \angle D & = & \angle B \\ \angle AXD & = & \angle AYB \ (=90^\circ) \\ AX & = & AY \end{array}\right\} \xrightarrow{AAS} \Delta AXD \cong \Delta AYB$$

$\xrightarrow[\text{sides}]{\text{corr.}} AB = AD.$

(12)



Suppose: $OA = OC = OB$; $OT = TX$, $OS = SY$, $OR = RZ$.

Show: (1) $\triangle ABC \cong \triangle XYZ$

(2) $YZ \parallel BC$, $XZ \parallel AC$, $XY \parallel AB$.

Proof: Outline.

(i) $\triangle ARO \cong \triangle BRO$

(ii) $ZBOA$ is a parallelogram

(iii) $ZBOA$ is a rhombus

(iv) symmetric argument $\Longrightarrow$ $COAY$ is a rhombus.

(v) $\left.\begin{array}{l} BZ = CY \\ BZ \parallel CY \end{array}\right\} \xrightarrow{1.8}$ $BZYC$ is a parallelogram

$\Longrightarrow \left\{\begin{array}{l} BC \parallel YZ \\ BC = YZ. \end{array}\right.$

(vi) symmetric argument $\Longrightarrow$ (a) $\left\{\begin{array}{l} AB \parallel XY \\ AB = XY \end{array}\right.$ (b) $\left\{\begin{array}{l} AC \parallel XZ \\ AC = XZ \end{array}\right.$

(vii) SSS $\Rightarrow \triangle ABC \cong \triangle XYZ$.

(i) $\underline{\triangle ARO}$ $\underline{\triangle BRO}$

$$\left.\begin{array}{lcl} AO & = & BO \\ RO & = & RO \end{array}\right\} \xrightarrow{HA} \boxed{\triangle ARO \cong \triangle BRO}$$

(ii) <u>ZBOA is a parallelogram.</u>

$\triangle ARO \cong \triangle BRO \xrightarrow[\text{sides}]{\text{corr.}} AR = BR$. But also, $OR = RZ$.

It follows that quadrilateral ZBOA has diagonals OZ and AB that bisect each other; $1.9 \Longrightarrow$ ZBOA is a parallelogram.

(iii) <u>ZBOA is a rhombus.</u>

Diagonals OZ and AB are perpendicular $\xrightarrow{10.4}$ ZBOA is a rhombus.

(iv) A symmetric argument $\Longrightarrow$ $\boxed{\text{COAY is also a rhombus.}}$

(v) <u>BZYC is a parallelogram.</u>

$$\left.\begin{array}{l} BZ = AO \\ AO = CY \end{array}\right\} \text{sides of rhombi ZBOA, COAY} \Longrightarrow BZ = CY.$$

We also have:

$$\left.\begin{array}{l} BZ \parallel AO \\ AO \parallel CY \end{array}\right\} \text{sides of rhombi} \Longrightarrow BZ \parallel CY$$

Now, we have $\left.\begin{array}{l} BZ = CY \\ BZ \parallel CY \end{array}\right\} \xrightarrow{1.8} BZYC$ is a parallelogram.

$\Longrightarrow \boxed{\left\{\begin{array}{l} BC = YZ \\ BC \parallel YZ \end{array}\right.}$

(vi) symmetric argument $\Longrightarrow$ $\boxed{\begin{array}{ll} (a) & \left\{\begin{array}{l} AB = XY \\ AB \parallel XY \end{array}\right. \\ (b) & \left\{\begin{array}{l} AC = XZ \\ AC \parallel XZ \end{array}\right. \end{array}}$

(vii) SSS $\Rightarrow \triangle ABC \cong \triangle XYZ$.