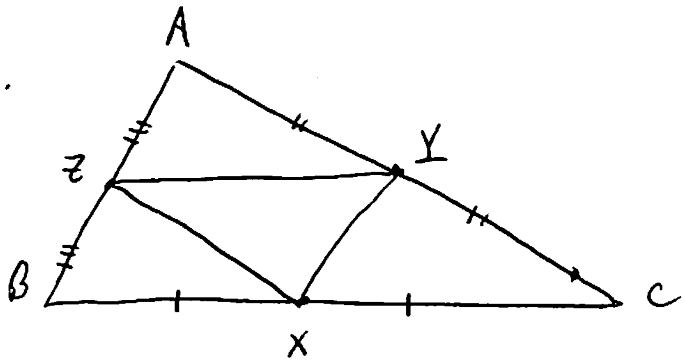
HW 4 Solutions

1H 1, 3, 4, 5, 6, 7, 8, 11

1H.1.



Show: $\Delta AYZ \cong \Delta YCX \cong \Delta ZXB \cong \Delta XZY$.

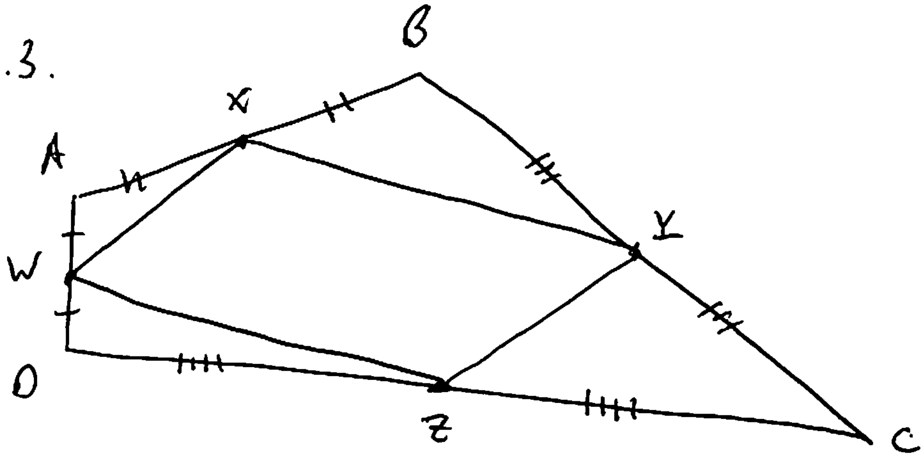
Proof: $\left.\begin{array}{l} Z = \text{midpt}(AB) \\ Y = \text{midpt}(AC) \end{array}\right\} \xRightarrow{1.31} ZY \parallel BC$ and $ZY = \frac{1}{2}BC = BX = XC$

Similarly, $1.31 \Longrightarrow XY \parallel AB$ and $XY = \frac{1}{2}AB$, $XZ \parallel AC$ and $XZ = \frac{1}{2}AC$

We have: $\left.\begin{array}{l} ZY = BX = XC \\ AZ = BZ = XY \\ AY = XZ = YC \end{array}\right\} \xRightarrow{SSS} \Delta AYZ \cong \Delta ZXB \cong \Delta YCX$

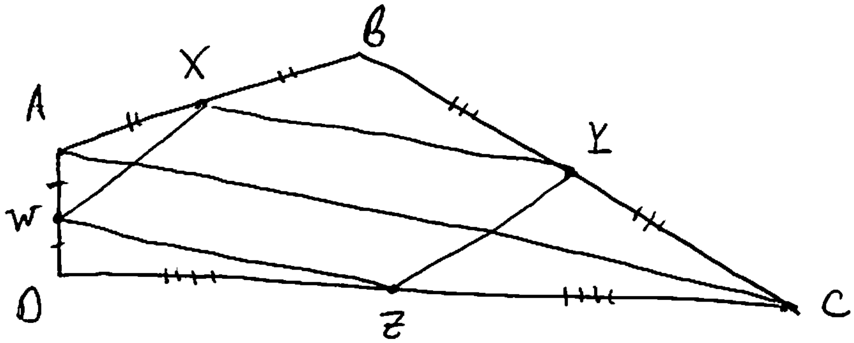
We also have: $\left.\begin{array}{l} XZ = AY \\ XY = AZ \\ YZ = YZ \end{array}\right\} \xRightarrow{SSS} \Delta AYZ \cong \Delta XZY.$

1.11.3.



<u>Show</u>: WXYZ is a parallelogram.

<u>Proof</u>:

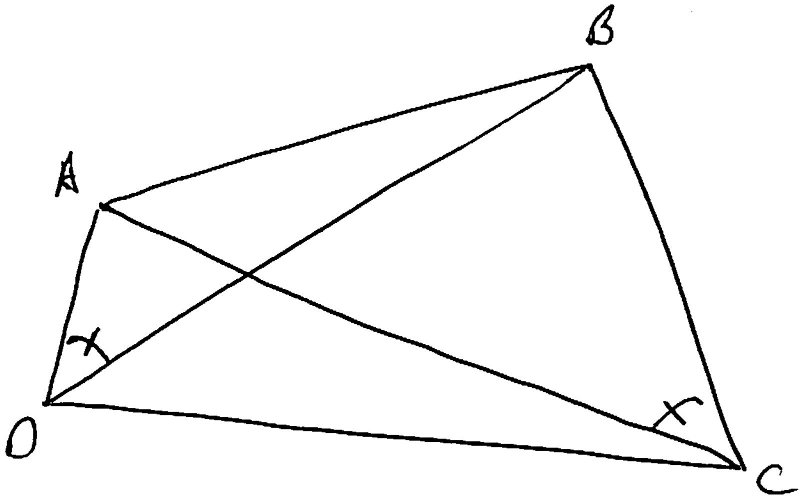


Apply 1.31 to
- ΔADC: $WZ \parallel AC$, $WZ = \frac{1}{2} AC$
- ΔABC: $XY \parallel AC$, $XY = \frac{1}{2} AC$

$\Rightarrow$ $WZ \parallel XY$ and $WZ = XY$

$\Rightarrow$ WXYZ is a parallelogram.

1H.4.



Suppose : $\angle ADB = \angle ACB$.

Show: $\angle ABD = \angle ACD$.

<u>Proof</u>: Consider $\underline{\triangle DXA}$ $\underline{\triangle CXB}$

$$\left.\begin{array}{l} \angle ADX = \angle BCX \text{ (given)} \\ \angle AXD = \angle BXC \text{ (vert. angles)} \end{array}\right\} \xRightarrow{AA} \triangle DXA \sim \triangle CXB$$

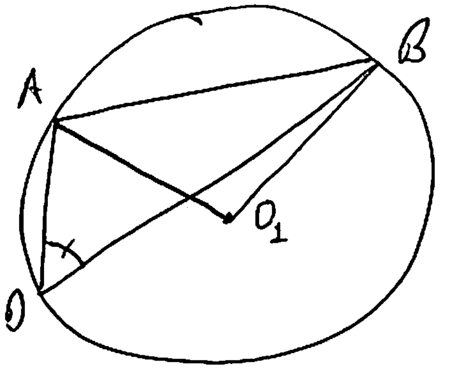
$$\xRightarrow[\text{parts}]{\text{corr.}} \frac{XC}{XD} = \frac{XB}{XA} \xRightarrow[\div XB]{\times XD} \frac{XD}{XA} = \frac{XC}{XB}.$$ Next, consider

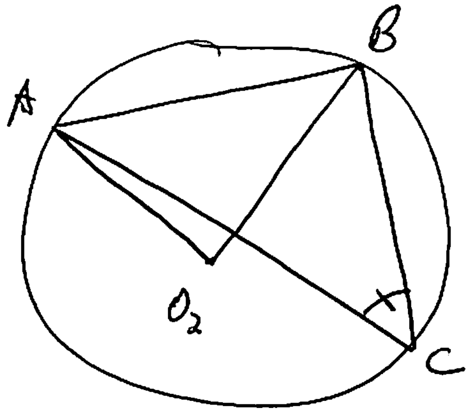
$\underline{\triangle XBA}$ $\underline{\triangle XCD}$

$$\left.\begin{array}{l} \angle AXB = \angle DXC \text{ (vert. angles)} \\ \dfrac{XD}{XA} = \dfrac{XC}{XB} \end{array}\right\} \xRightarrow{SAS} \triangle XBA \sim \triangle XCD$$

$$\xRightarrow[\text{parts}]{\text{corr}} \angle ABD = \angle ACD.$$

Alternative proof using circles.





S_1 : circumcircle of ΔABD $\qquad$ S_2 : circumcircle of ΔABC.

In S_1, $\angle ADB \doteq \frac{1}{2}\widehat{AB}$ $\qquad$ In S_2, $\angle ACB \doteq \frac{1}{2}\widehat{AB}$

$= \angle AO_1B.$ $\qquad$ $= \angle AO_2B.$

$\Longrightarrow \boxed{\angle AO_1B = \angle AO_2B}$ To show: $\Delta AO_2B \cong \Delta AO_2B$.

$$\left.\begin{array}{l} \text{In } \Delta AO_2B,\ AO_1 = BO_1 \xrightarrow{\text{p.a}} \angle O_1AB = \angle O_1BA \\ \text{In } \Delta AO_2B,\ AO_2 = BO_2 \xrightarrow{\text{p.a}} \angle O_2AB = \angle O_2BA \end{array}\right\} \Longrightarrow$$

In ΔAO_1B, $\angle AO_1B + \angle O_1AB + \angle O_1BA = \angle AO_1B + 2\angle O_1AB = 180°$

In ΔAO_2B, $\angle AO_2B + \angle O_2AB + \angle O_2BA = \angle AO_2B + 2\angle O_2AB = 180°$.

$$\Longrightarrow \angle AO_1B + 2\angle O_1AB = 180° = \angle AO_2B + 2\angle O_2AB \xRightarrow{\angle AO_1B = \angle AO_2B} \boxed{\angle O_1AB = \angle O_2AB}$$

We now have: $\underline{\Delta AO_1B}$ $\qquad$ $\underline{\Delta AO_2B}$

$$\left.\begin{array}{lcl} \angle AO_1B & = & \angle AO_2B \\ \angle O_1AB & = & \angle O_2AB \\ AB & = & AB \end{array}\right\} \begin{array}{l} \xRightarrow{AAS} \Delta AO_1B \cong \Delta AO_2B \\ \xRightarrow[\text{parts}]{\text{corr.}} AO_1 = AO_2 \end{array}$$

$\xrightarrow[\text{parts}]{\text{corr}} AO_1 = AO_2, \ \angle O_1AB = \angle O_2AB.$

Observe: $\left.\begin{array}{l}\angle O_1AD = \angle BAD - \angle O_1AB \\ \angle O_2AD = \angle BAD - \angle O_2AB\end{array}\right\} \Longrightarrow \angle O_1AO = \angle O_2AD$

Consider $\underline{AO_1D}$ $\underline{AO_2D}$

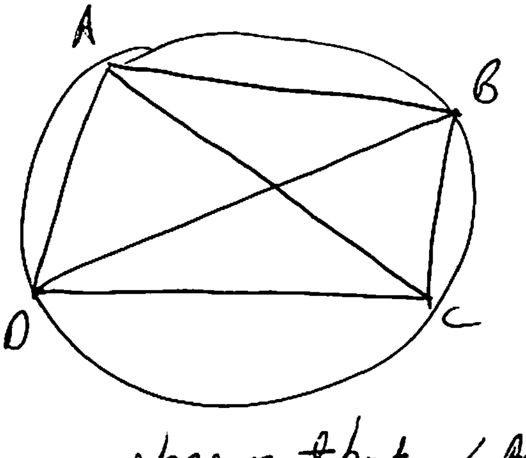
$$\left.\begin{array}{lcl} AO_1 & = & AO_2 \\ \angle O_1AD & = & \angle O_2AD \\ AD & = & AD \end{array}\right\} \xRightarrow{SAS} \Delta AO_2D \cong \Delta AO_2D$$

$\xrightarrow[\text{parts}]{\text{corr.}} DO_1 = DO_2$. We now have: $\underbrace{DO_2 = DO_1 = AO_1}_{}$ $= AO_2$

A, D both on S_1

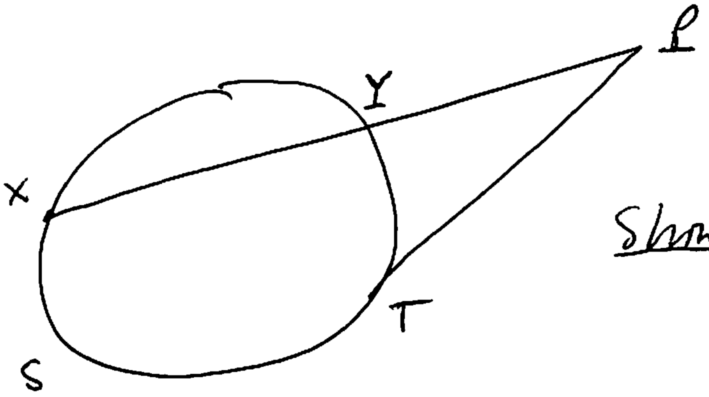
$\Longrightarrow$ D is on the circle with radius AO_2, namely S_2.

$\Longrightarrow$ A, B, C, D are on S_2 (ABCD is cyclic).



To conclude, we observe that $\angle ACD \overset{\circ}{=} \frac{1}{2}\widehat{AD} \overset{\circ}{=} \angle ABD$.

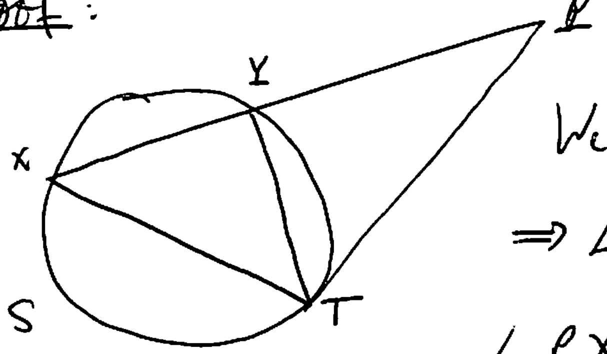
14.5.



Suppose that PT is tangent to S.

Show: $(PX)(PY) = (PT)^2$.

Proof:

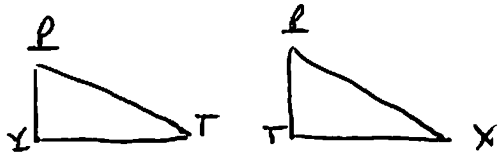


We will show that $\Delta PYT \sim \Delta PTX$.

We observe: PT tangent to S

$\Rightarrow \angle PTY \overset{\circ}{=} \frac{1}{2} \widehat{YT}$. But also, we have

$\angle PXT \overset{\circ}{=} \frac{1}{2} \widehat{YT}$, so $\angle PTY = \angle PXT$.

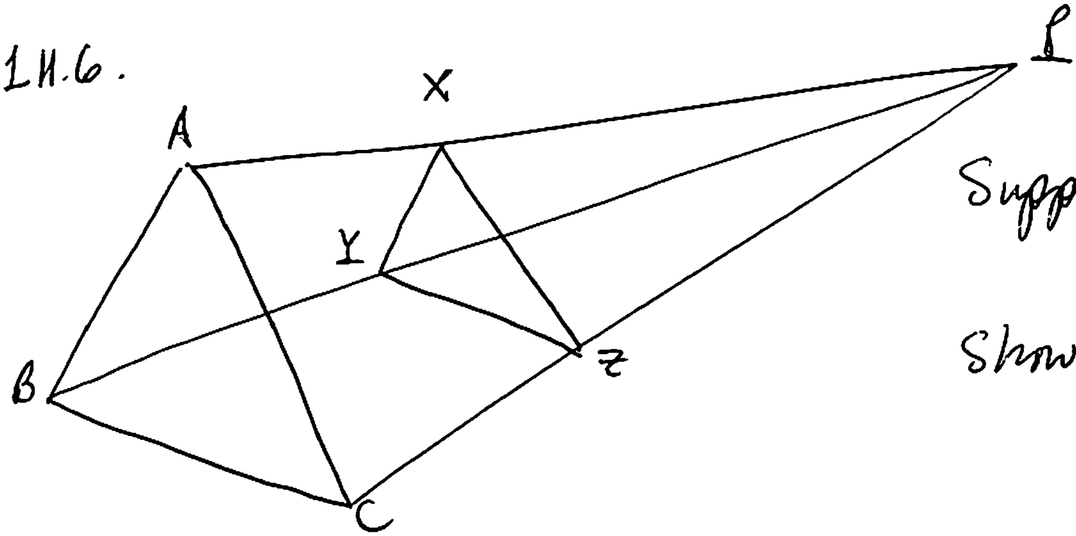


Consider:

$$\begin{array}{ccc} \underline{\Delta PYT} & & \underline{\Delta PTX} \\ \angle P & = & \angle P \\ \angle PTY & = & \angle PXT \end{array} \Bigg\} \xRightarrow{AA} \Delta PYT \sim \Delta PTX$$

$$\xRightarrow[\text{parts}]{\text{corr}} \frac{PY}{PT} = \frac{PT}{PX} \xRightarrow[\times PT]{\times PX} (PX)(PY) = (PT)^2.$$

1H.6.



Suppose: $XY \parallel AB$,
$YZ \parallel BC$.

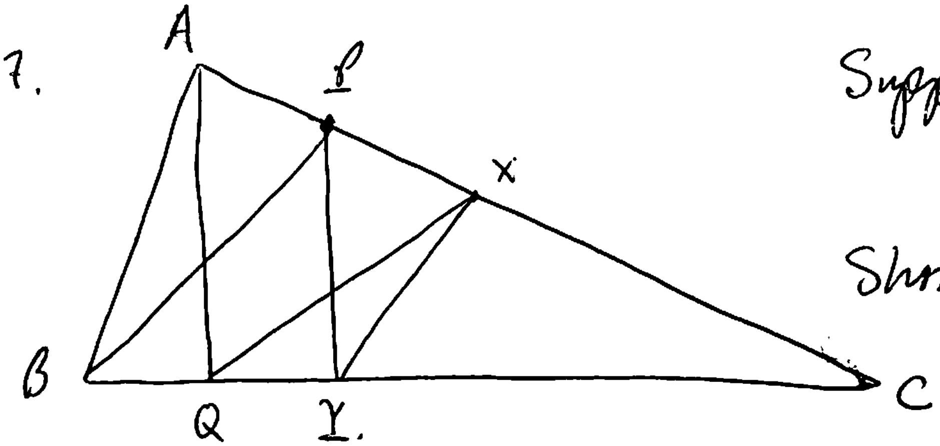
Show: $XZ \parallel AC$.

<u>Proof</u>: Apply 1.29 in

$$\left.\begin{array}{l} \Delta ABP : XY \parallel AB \Rightarrow \dfrac{PX}{PA} = \dfrac{PY}{PB} \\ \Delta BCP : YZ \parallel BC \Rightarrow \dfrac{PZ}{PC} = \dfrac{PY}{PB} \end{array}\right\} \Rightarrow \frac{PX}{PA} = \frac{PZ}{PC}$$

$$\xRightarrow[\text{in } \Delta ACP]{1.29} XZ \parallel AC.$$

1H.7.



Suppose: $PY \parallel AQ$,
$QX \parallel BP$.

Show: $XY \parallel AB$.

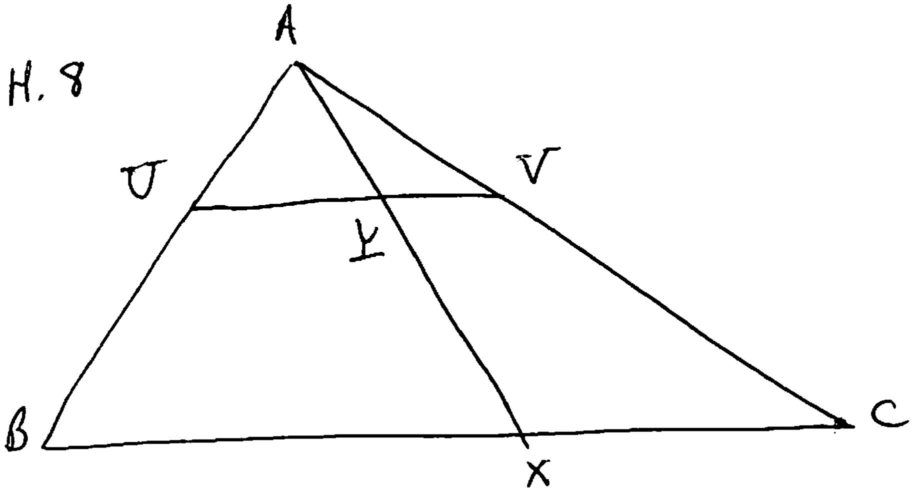
__Proof:__

Apply 1.29 to ΔCPB: $QX \parallel BP \Rightarrow \dfrac{CX}{CP} = \dfrac{CQ}{CB} \xrightarrow[\times CB]{\times CP} (CX)(CB) = (CP)(CQ)$

ΔCAQ: $PY \parallel AQ \Rightarrow \dfrac{CP}{CA} = \dfrac{CY}{CQ} \xrightarrow[\times CQ]{\times CA} (CP)(CQ) = (CA)(CY)$

$\Rightarrow (CX)(CB) = (CA)(CY) \xrightarrow[\div CB]{\div CA} \dfrac{CX}{CA} = \dfrac{CY}{CB} \xrightarrow[\text{in } \Delta ABC]{1.29} XY \parallel AB.$

LH. 8



Suppose: $UV \parallel BC$

Show: $\dfrac{UY}{YV} = \dfrac{BX}{XC}$.

Proof: Consider

① $\triangle AUY$ $\quad$ $\triangle ABX$

$$\left.\begin{array}{l} \angle UAY = \angle BAX \text{ (same angle)} \\ \angle AUY = \angle ABX \text{ (corr. angles)} \end{array}\right\} \xrightarrow{AA} \triangle AUY \sim \triangle ABX$$

$$\xrightarrow[\text{parts}]{\text{corr}} \frac{AY}{AX} = \frac{UY}{BX}$$

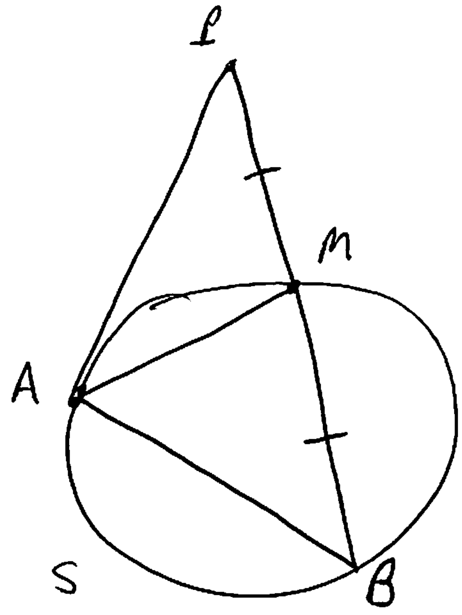
② $\triangle AYV$ $\quad$ $\triangle AXC$

$$\left.\begin{array}{l} \angle YAV = \angle XAC \text{ (same angle)} \\ \angle AYV = \angle AXC \text{ (corr. angles)} \end{array}\right\} \xrightarrow{AA} \triangle AYV \sim \triangle AXC$$

$$\xrightarrow[\text{parts}]{\text{corr.}} \frac{AY}{AX} = \frac{YV}{XC}.$$

$$\longrightarrow \frac{UY}{BX} = \frac{YV}{XC} \xrightarrow[\times BX]{\div YV} \frac{UY}{YV} = \frac{BX}{XC}.$$

1H. 11



Suppose: PA is tangent to S, $AM = 1$, and $M = \text{midpt}(PB)$ (so $PM = MB$).

Find AB.

Solution: Same reasoning as in 1H.5. shows that $\triangle PMA \sim \triangle PAB$.

$$\xRightarrow[\text{pnts}]{\text{corr.}} \frac{AB}{AM} = \frac{PB}{PA} = \frac{PA}{PM}. \quad \text{Note: } PM = MB \Rightarrow PM = \frac{1}{2} PB.$$

$$\therefore \frac{PB}{PA} = \frac{PA}{PM} \xrightarrow[\times PM]{\times PA} (PM)(PB) = (PA)^2 \xRightarrow[=\frac{1}{2}PB]{PM} \frac{1}{2}(PB)^2 = (PA)^2$$

$$\Longrightarrow \underbrace{PB = \sqrt{2}(PA)}$$

$$\therefore AB \underset{\substack{\uparrow \\ AM=1}}{=} \frac{AB}{AM} = \frac{PB}{PA} = \frac{\sqrt{2}(PA)}{PA} = \sqrt{2}. \quad \text{I.e., } \boxed{AB = \sqrt{2}}$$