Midterm Examination 1 - Math 544, Frank Thorne (thorne@math.sc.edu)

Wednesday, October 13, 2015

Please work without books, notes, calculators, or any assistance from others.

(1) (15 points) Consider the line $\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -5 \\ 4 \end{bmatrix}$, or to say the same thing another way, the line

$$\left\{ t \begin{bmatrix} -5 \\ 4 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

Find a linear equation of the line and a vector in $\mathbb{R}^2$ which is orthogonal to every vector on the line. Draw a picture which illustrates your conclusions.

(2) (15 points)

(a) Compute the plane through the points $(2, 1, 2)$, $(3, 0, 4)$, and $(5, 1, 3)$ in $\mathbb{R}^3$. (Write your answer as a set.) Draw a schematic diagram which illustrates your answer.

(b) Find any point other than the three above which is on this plane.

(3) (24 points) Determine, with proof, whether each of the following subsets of $\mathbb{R}^4$ is a subspace of $\mathbb{R}^4$.

(a)

$$Y = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; x = w \text{ and } z = -y \right\}$$

(b)

$$Y = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; x = w \text{ or } z = w \right\}$$

(4) (20 points) Determine whether or not the equation

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

has a nontrivial solution. Show your work (which should include row reducing an augmented matrix).

(5) (26 points) Let P_3 be the vector space consisting of all polynomials of degree at most 3. Let S be the subset

$$S = \{1 + t^2,\ 2 + 2t,\ 3t^2 - 3t\}.$$

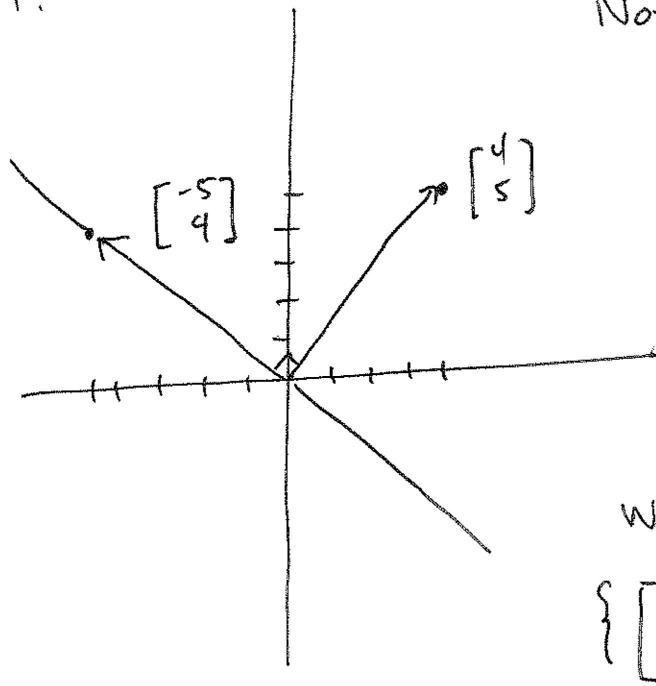
(a) Is S a subspace of P_3? Why or why not?

(b) Describe explicitly the set Span(S). (*Hint: This is very easy if you understand the definition.*)

(c) Determine whether S is linearly dependent or linearly independent.

(d) Is $t^3 \in \text{Span}(S)$? Why or why not?

1.



Note that $\begin{bmatrix}4\\5\end{bmatrix} \cdot \begin{bmatrix}-5\\4\end{bmatrix} = 4(-5) + 5 \cdot 4 = 0.$

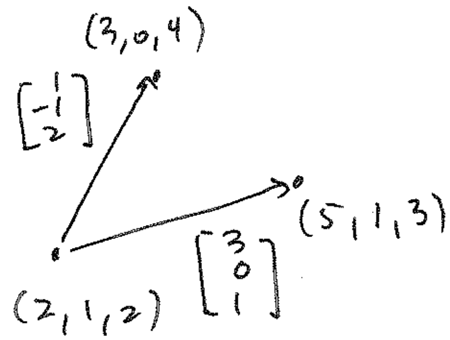
So $\begin{bmatrix}4\\5\end{bmatrix}$ is orthogonal to $\begin{bmatrix}-5\\4\end{bmatrix}$ and every multiple of $\begin{bmatrix}-5\\4\end{bmatrix}$.

We can write the line as

$$\left\{ \begin{bmatrix}x\\y\end{bmatrix} : \begin{bmatrix}x\\y\end{bmatrix} \cdot \begin{bmatrix}4\\5\end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix}x\\y\end{bmatrix} : 4x + 5y = 0 \right\}.$$

2.



Choosing (2,1,2) as a base point vectors to the other two points are $\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}3\\0\\4\end{bmatrix} - \begin{bmatrix}2\\1\\2\end{bmatrix}$

and $\begin{bmatrix}3\\0\\1\end{bmatrix} = \begin{bmatrix}5\\1\\3\end{bmatrix} - \begin{bmatrix}2\\1\\2\end{bmatrix}$.

So the plane is

$$\left\{ \begin{bmatrix}2\\1\\2\end{bmatrix} + r\begin{bmatrix}1\\-1\\2\end{bmatrix} + s\begin{bmatrix}3\\0\\1\end{bmatrix} : r, s \in \mathbb{R} \right\}.$$

(Many other formulations are also correct.)

A point on it is $\begin{bmatrix}2\\1\\2\end{bmatrix} + \begin{bmatrix}1\\-1\\2\end{bmatrix} + \begin{bmatrix}3\\0\\1\end{bmatrix} = \begin{bmatrix}6\\0\\5\end{bmatrix}$.

(There are many others.)

3. $Y = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 : x = w \text{ and } z = -y \right\}$ is a subspace.

We must check:

(1) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in Y$ because $0 = 0$ and $0 = -0$.

(2) If $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in Y$ and $r \in \mathbb{R}$, then $r \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ rz \\ rw \end{bmatrix}$

is in Y because $rx = rw$ and $ry = -rz$ (if $x = w$ and $y = -z$).

(3) Assume $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} \in Y$. Their sum is $\begin{bmatrix} x + x' \\ y + y' \\ z + z' \\ w + w' \end{bmatrix}$.

This is in Y: Because $x = w$ and $x' = w$, $x + x' = w + w'$, because $z = -y$ and $z' = -y'$, $z + z' = -(y + y')$.

(b) $Y = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 : x = w \text{ or } z = w \right\}$ is not a subspace,

because $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ are both in Y but their

sum $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ is not.

4. We must solve

$$\begin{bmatrix} x_1 + x_2 + x_3 \\ 2x_1 + 0x_2 - x_3 \\ 2x_1 - x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

i.e.

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 + 0x_2 - x_3 &= 0 \\ 2x_1 - x_2 + 2x_3 &= 0. \end{aligned}$$

The associated augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 0 & -1 & 0 \\ 2 & -1 & 2 & 0 \end{array}\right].$$

Row reducing:

Sub 2·R1 from R2
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 0 \\ 2 & -1 & 2 & 0 \end{array}\right]$$

Sub 2·R1 from R3
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & -3 & 0 & 0 \end{array}\right]$$

Mul R3 by $\frac{-1}{3}$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Switch R2, R3
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & -3 & 0 \end{array}\right]$$

Sub R2 from R1
Add 2·R2 to R3
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

Mul R3 by $\frac{-1}{3}$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Sub R3 from R1
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

In other words, our equation has <u>only</u> the <u>trivial solution</u>.

5.

(a) No, (for example) because it doesn't contain 0.

(b)

$$\mathrm{Span}(S) = \left\{ a(1+t^2) + b(2+2t) + c(3t^2-3t) : a,b,c \in \mathbb{R} \right\}$$

(c: answer #1)

S is linearly <u>dependent</u> because

$$3t^2 - 3t = 3(1+t^2) - \frac{3}{2}(2+2t).$$

(c: answer #2)

S is linearly <u>dependent</u> because

$$3(1+t^2) - \frac{3}{2}(2+2t) - 1 \cdot (3t^2 - 3t) = 0,$$

a nontrivial linear combination equaling zero.

(Other formulations are also possible.)

(d) $t^3 \notin \mathrm{Span}(S)$. Every element of S has t^3 coefficient zero, so the same will be true of linear combinations of elements of S. So $\mathrm{Span}(S)$ does not contain t^3 or any other polynomial with t^3 coefficient nonzero.