Midterm Examination 2 - Math 544, Frank Thorne (thorne@math.sc.edu)

Monday, November 20

Twenty points for each question. Please work without books, notes, calculators, or any assistance from others.

(1) Suppose that V is a vector space, $S = \{v_1, v_2, \cdots, v_n\}$ is a finite list of vectors in V, and W is a subspace of V. Define the following terminology:

7 (a) The **span** of S;

6 (b) what it means for S to be **linearly independent**;

6 (c) what it would mean for S to be a **basis** of W;

6 (d) the **dimension** of W.

(2) The matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

represents a linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$.

5 (a) Compute $T_A\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right)$.

10 (b) Compute the rank, nullity, and kernel of A. (You should be able to do this in your head.)

10 (c) By means of a description and/or a cartoon, describe T_A sufficiently well such that someone else could visualize it.

(3) Each day, a USC student chooses a meat lunch or a vegetarian lunch. Assume that her choices are modeled by a Markov chain where each day's choice is based on the previous choice. Overall this student prefers meat, but occasionally eats vegetarian, especially if she hasn't done so recently.

8 (a) If M is a transition matrix matching the above description, then which of these could M be?

$$M_1 = \begin{bmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.9 & 0.6 \\ 0.1 & 0.4 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0.9 & 0.1 \\ 0.6 & 0.4 \end{bmatrix}.$$

Explain why.

7 (b) Compute M^2, and briefly describe what it is telling you.

10 (c) Compute a steady state vector for the Markov chain.

Removed. Good for bonus

~~(4)~~ Prove that a linear transformation T is one-to-one if and only if $T(\vec{0}) = \vec{0}$.

Note. The elements $\vec{0}$ on the left and right refer to the identity element of the domain and target (codomain) vector spaces, and don't necessarily equal each other.

(5) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}.$$

Compute bases for Row(A), Col(A), Ker(A), and compute Rank(A) and Nullity(A).

(5 each)

1. (a) The span of S is the set of all linear combinations

$$\{a_1\vec{v}_1 + \cdots a_n\vec{v}_n : a_1, \dots, a_n \in \mathbb{R}\}.$$

(b) This means that if

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0}$$

then all the a_i's are zero. (Other equivalent definitions are also correct.)

(c) This means that S spans W and is linearly independent.

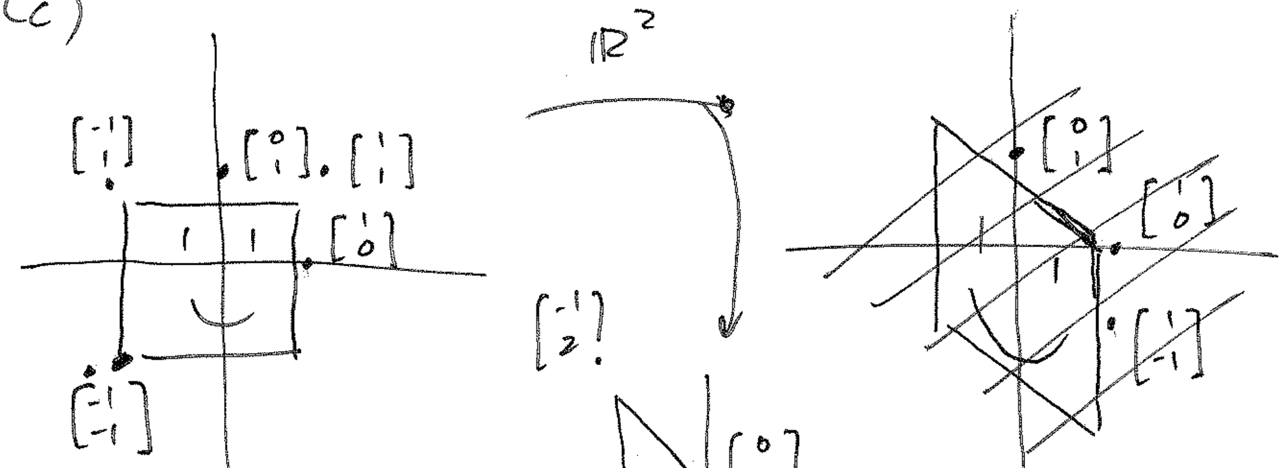
(d) The number of vectors in any basis of W.

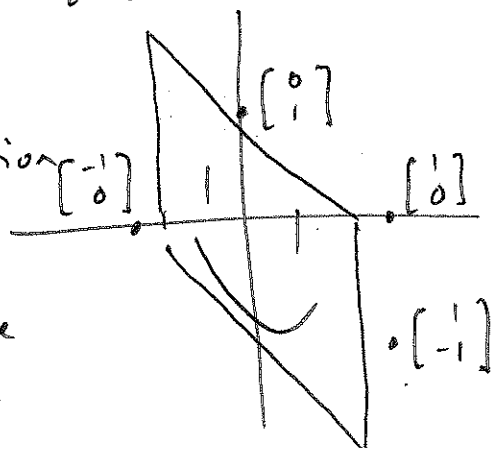
2. (a) $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 0\cdot 5 \\ -1\cdot 2 + 1\cdot 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$

(b) The rank is 2 because the columns are visibly independent.

Since rank + nullity = # of columns = 2, the nullity is 0 and the kernel is $\{\vec{0}\}$.

(c)





This linear transformation represents a "shear" where the right is pulled down and the left is pulled up.

3.

$$
\begin{array}{cccc}
 & & \text{meat} & \text{veg} \\
M = M_1 = & \left[\begin{array}{c} \\ \\ \end{array}\right. & \begin{array}{c} 0.6 \\ 0.4 \end{array} & \begin{array}{c} 0.9 \\ 0.1 \end{array} \left.\begin{array}{c} \\ \\ \end{array}\right] \begin{array}{l} \text{meat} \\ \text{veg} \end{array}
\end{array}
$$

If she eats meat, she is 40% likely to eat veg tomorrow, but if she eats veg she will almost certainly eat meat tomorrow. She is more likely to eat meat no matter what.

M_2 says the student is almost certain to eat meat tomorrow if she did so today, but is more likely to eat veg if she did so today — not in accordance with the problem.

In M_3 the columns don't add to 1!

$$
M^2 = \begin{bmatrix} 0.6 \cdot 0.6 + 0.9 \cdot 0.4 & 0.6 \cdot 0.9 + 0.9 \cdot 0.1 \\ 0.4 \cdot 0.6 + 0.1 \cdot 0.4 & 0.4 \cdot 0.9 + 0.1 \cdot 0.1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0.72 & 0.63 \\ 0.28 & 0.37 \end{bmatrix}
$$

This describes what the student will do two days from now, based on her choice today. She is more likely to eat meat if she did so today.

Solve $M\vec{v} = I\vec{v}$, so $(M - I)\vec{v} = 0$

$$M - I = \begin{bmatrix} -0.4 & 0.9 \\ 0.4 & -0.9 \end{bmatrix} \xrightarrow[\text{by } -10]{\text{Mul everything}} \begin{bmatrix} 4 & -9 \\ -4 & 9 \end{bmatrix}$$

~~so~~ Add R1 to R2 $\begin{bmatrix} 4 & -9 \\ 0 & 0 \end{bmatrix}$

Mul R1 by $\frac{1}{4}$ $\begin{bmatrix} 1 & -9/4 \\ 0 & 0 \end{bmatrix}$

The nullspace (kernel) of $M - I$ is $\left\{ \begin{bmatrix} 9/4\, r \\ r \end{bmatrix} : r \in \mathbb{R} \right\}$.

If these add to 1 then $\frac{13}{4} r = 1$ so $r = \frac{4}{13}$,

and a steady ~~state~~ vector is $\begin{bmatrix} 9/13 \\ 4/13 \end{bmatrix}$.

4. Question doesn't count! (oops) Bonus if you answered as follows:

The claim is false. It is always true that $T(\vec{0}) = \vec{0}$ for any linear transformation. Therefore "only if" is vacuously true, but "if" is not. For example, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ represents a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ for which $T(\vec{0}) = \vec{0}$, which is not one-to-one.

5.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix} \xrightarrow[\text{from R2}]{\text{Sub 2R1}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow[\text{to R3}]{\text{Add 2R2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rows are visibly linearly independent (always true in RREF) so a basis for Row(A) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ and Rank(A) = 2.

Since Col(A) has dimension 2 also a basis is given by any two linearly independent vectors, for example

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

Nullity(A) = 3 - Rank(A) = 1.

We read off the kernel from the RREF basis above,

$$\text{Ker}(A) = \left\{ \begin{bmatrix} -r \\ -r \\ r \end{bmatrix} : r \in \mathbb{R} \right\} \text{ and a basis is } \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$