Linear Transformations, and matrices.

Def. (p. 400) Let V and U be two vector spaces. A function $T: V \to U$ is a <u>linear transformation</u> if, for all $\vec{v_1}, \vec{v_2} \in V$ and $c \in V$, we have

$$T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2})$$

$$T(c\vec{v_1}) = c\,T(\vec{v_1}).$$

<u>Examples with $\mathbb{R}^2$.</u>

Define $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$

T_2

$$\left.\begin{aligned} T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} x \\ y \end{bmatrix} \\ T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ T_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} y \\ x \end{bmatrix}. \\ T_4\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} y \\ -x \end{bmatrix}. \end{aligned}\right\} \text{Draw pictures on board.}$$

$$T = T_5\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \cancel{\tfrac{\sqrt{3}}{2}x + \tfrac{1}{2}x} \begin{bmatrix} \frac{\sqrt{3}}{2}x - \frac{1}{2}y \\ \frac{1}{2}x + \frac{\sqrt{3}}{2}y \end{bmatrix}.$$

Can we draw a picture?

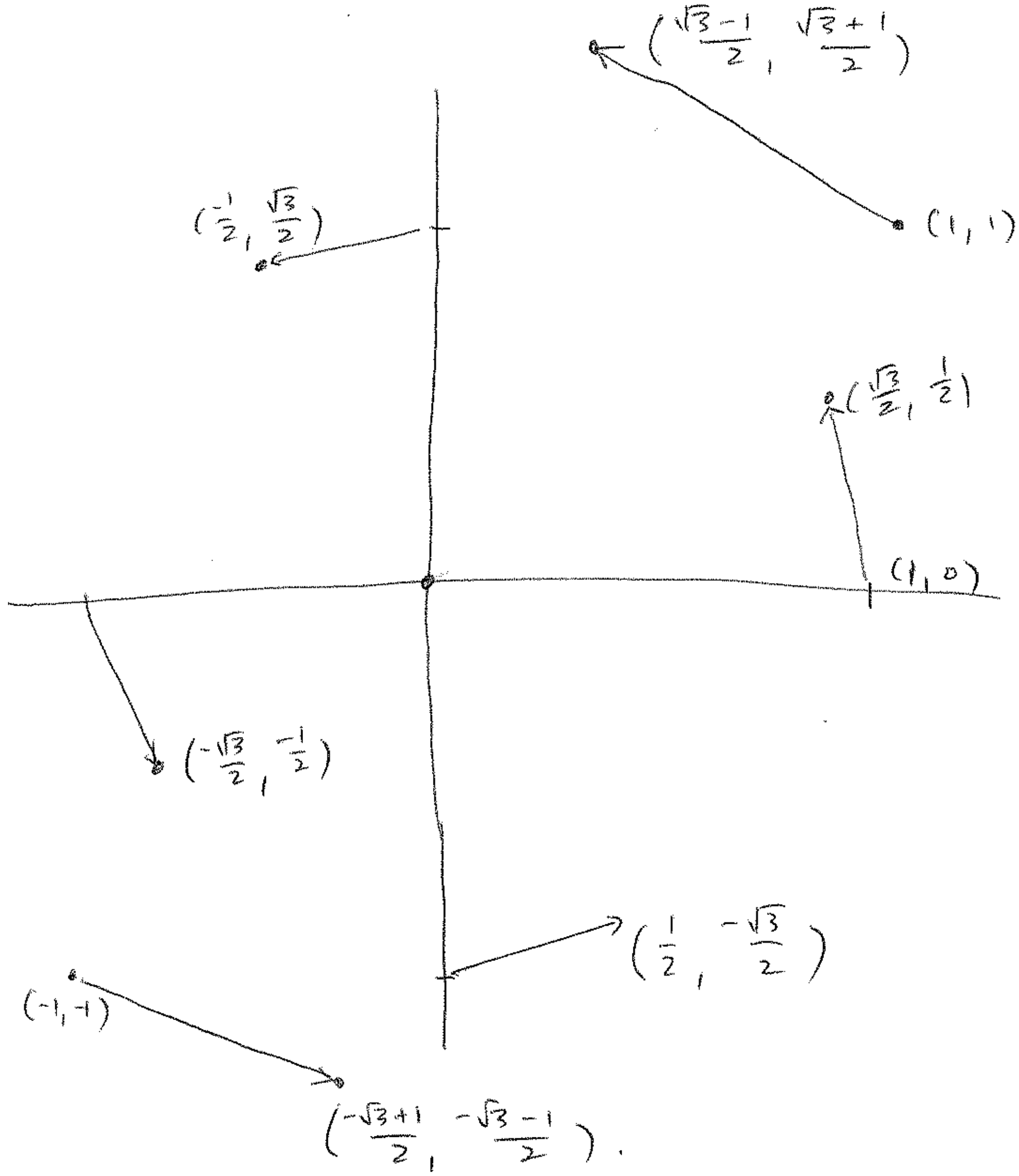
$$T_5\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

$$T_5\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

$$T_5\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix}, \quad T_5\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

$$T_5\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T_5\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T_5\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{\sqrt{3}+1}{2} \end{bmatrix}$$

T



~~Rotations Suppose~~

Proposition. T is rotation by 30 degrees.

Proof. Need to check two things:

(1) $|T(\vec{v})| = |\vec{v}|$ for all $\vec{v} \in \mathbb{R}^2$.

(2) $T(\vec{v}) \cdot \vec{v} = |T(\vec{v})| \cdot |\vec{v}| \cos(30^\circ)$.

If $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, $|\vec{v}|^2 = x^2 + y^2$.

$$|T(\vec{v})|^2 = \left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)^2 + \left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)^2$$

$$= \frac{3}{4}x^2 - \frac{\sqrt{3}}{2}xy + \frac{1}{4}y^2 + \frac{1}{4}x^2 + \frac{\sqrt{3}}{2}xy + \frac{3}{4}y^2$$

$$= x^2 + y^2.$$

(2) $T(\vec{v}) \cdot \vec{v} = x \cdot \left(\frac{\sqrt{3}}{2} x - \frac{1}{2} y \right) + y \cdot \left(\frac{1}{2} x + \frac{\sqrt{3}}{2} y \right)$

$= \frac{\sqrt{3}}{2} x^2 - \frac{1}{2} xy + \frac{1}{2} xy + \frac{\sqrt{3}}{2} y^2 = \frac{\sqrt{3}}{2} (x^2 + y^2).$

Exercise. Do the same for

T: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{3}}{2} x + \frac{1}{2} y \\ -\frac{1}{2} x + \frac{\sqrt{3}}{2} y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} y \\ \frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}.$

Figure out what angle rotation they are and prove it in the same manner.

Extra Credit. Figure out the pattern and write down a matrix representing rotation by θ.

Example. Do the same for $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix}.$

Proposition. Linear transformations : $\mathbb{R}^2 \to \mathbb{R}^2$ are those which we can write of the form

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \quad \text{for some } a, b, c, d \in \mathbb{R}.$$

Notation. The associated matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Write $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ for above.

Observations.

(1) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is any linear transformation,

$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(2) what if T sends every vector to 0?

Clearly, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0x + 0y \\ 0x + 0y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Conversely: Suppose $\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$ for all $\begin{bmatrix} x \\ y \end{bmatrix}$.

Then plug in $x = 1, y = 0$: $a = c = 0$.

Plug in $x = 0, y = 1$: $b = d = 0$.

(3) Suppose $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ are nonzero and parallel,

i.e. $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \vec{w}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \lambda \vec{w}$ for some $\lambda \in \mathbb{R}$.

Then $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + y\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$= x \cdot \vec{w} + y \cdot \lambda \vec{w} = (x + y\lambda)\,\vec{w}$.

This means the image of T lies on a line

(4) Suppose $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ are linearly independent.

Now $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = x\,T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + y\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$.

Three observations.

* what if this is 0? BY HYPOTHESIS $x = y = 0$.

  So then $T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = 0$ and $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) \neq 0$ if $\begin{bmatrix}x\\y\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}$

* Saw before, $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ span $\mathbb{R}^2$.

  This means, the range of T is all of $\mathbb{R}^2$.

* Now, suppose $T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right) = T\left(\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)$.

  Then
  $$T\left(\begin{bmatrix}x_1-x_2\\y_1-y_2\end{bmatrix}\right) = T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right) + T\left(\begin{bmatrix}-x_2\\-y_2\end{bmatrix}\right) = T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right) - T\left(\begin{bmatrix}x_2\\y_2\end{bmatrix}\right) = 0.$$

  By linear independence, $\left.\begin{matrix}x_1 - x_2 = 0\\ y_1 - y_2 = 0\end{matrix}\right\}$ i.e. $x_1 = x_2$, $y_1 = y_2$.

This means T is one-to-one and onto.

If T has matrix $\begin{bmatrix}a & b\\c & d\end{bmatrix}$, its inverse has matrix
$$\frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}.$$