Quiz 5 - Math 544, Frank Thorne (thorne@math.sc.edu)

Monday, October 12, 2015

1. What does it mean for a set of vectors S to be linearly dependent? You may give the definition, or answer using any of the 'if and only if' results from lecture or the book.

2. Determine whether each of the following subsets of $\mathbb{R}^2$ is linearly dependent or not.

$$S_1 := \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right\}$$

$$S_2 := \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$$

1. Three correct answers:

(a) There is some $\vec{v} \in S$ with $\mathrm{Span}(S - \{\vec{v}\}) = \mathrm{Span}(S)$.

(b) $S = \{\vec{0}\}$ or some element of S is a linear combination of the others.

(Note: Contrary to what I said in lecture, you don't need to specify "nontrivial". If $\vec{v} = 0 \cdot \vec{w}_1 + \cdots + 0 \cdot \vec{w}_k$ for some vectors $\vec{w}_1, \ldots, \vec{w}_k$, then $\vec{v} = \vec{0}$ and $\mathrm{Span}(S - \{\vec{v}\}) = \mathrm{Span}(S)$.)

(c) You can write

$$a_1 \vec{v}_1 + \cdots + a_k \vec{v}_k = 0$$

for some vectors $\vec{v}_1, \ldots, \vec{v}_k$ and scalars $a_1, \ldots a_k$ where at least one of the scalars is not zero.

2. $S_1 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right\}$ is linearly dependent.

To see this by (three correct answers:)

(a): We already saw that any two nonzero, nonparallel vectors in $\mathbb{R}^2$ span all of $\mathbb{R}^2$. So, any two vectors in S_1 span $\mathbb{R}^2$. S_1 also spans $\mathbb{R}^2$ because it is as least as large as the span of any subset of S_1, but contained within $\mathbb{R}^2$.

So, if $\vec{v}$ is any vector in S_1, $\mathrm{Span}(S_1 - \{\vec{v}\}) = \mathrm{Span}(S_1)$

(b) We can write $\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) $1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ is linearly independent.

Proof using (b).

If $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a linear combination of the others then $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = a \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ for some a. So $a = \frac{-1}{2}$ and $a = \frac{2}{3}$, impossible.

Similarly, if $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ is a linear combination of the others then $\begin{bmatrix} -2 \\ 3 \end{bmatrix} = b \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for some $b \in \mathbb{R}$. Then $b = -2$ and $b = \frac{3}{2}$, impossible.

Alternative proof using (c).

If $a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

we can solve
$$a_1 - 2a_2 = 0$$
$$2a_1 + 3a_2 = 0.$$

The associated augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & 3 & 0 \end{array}\right] \xrightarrow[\text{from } R2]{\text{Subtract } 2\cdot R1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 7 & 0 \end{array}\right]$$

$$\xrightarrow[R2 \text{ by } 1/7]{\text{Multiply}} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[\text{to } R1]{\text{Add } 2\cdot R2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

So $a_1 = 0$, $a_2 = 0$ is the only solution.

This means that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ are linearly independent.

4.1, A5.

Let $U = \left\{ \begin{bmatrix} a & b & a+b \\ c & a+c & b+c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.

Find a finite set of vectors that spans U.

We can write

$$U = \left\{ a\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\},$$

so by definition

$$\operatorname{Span}\left(\left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right\} \right) = U.$$

4.1, B2.

Let $U = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : x + z = 0 \text{ and } y + w = 0 \right\}$.

Find a finite set of vectors that spans U.

We can write these equations as a matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

which is already in RREF. We see that z and w are free variables, $x = -z$, $y = -w$, so

$$U = \left\{ r\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} : r, s \in \mathbb{R} \right\},$$

i.e. $\operatorname{Span}\left(\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \right) = U.$