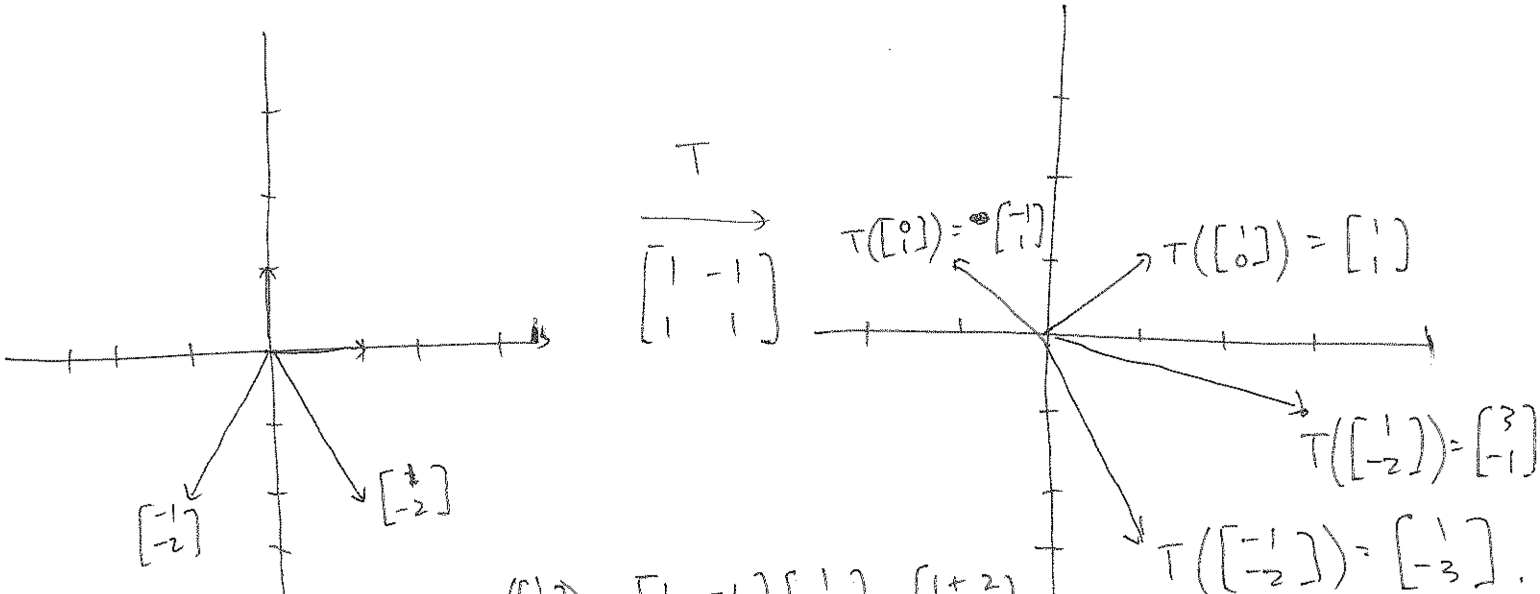
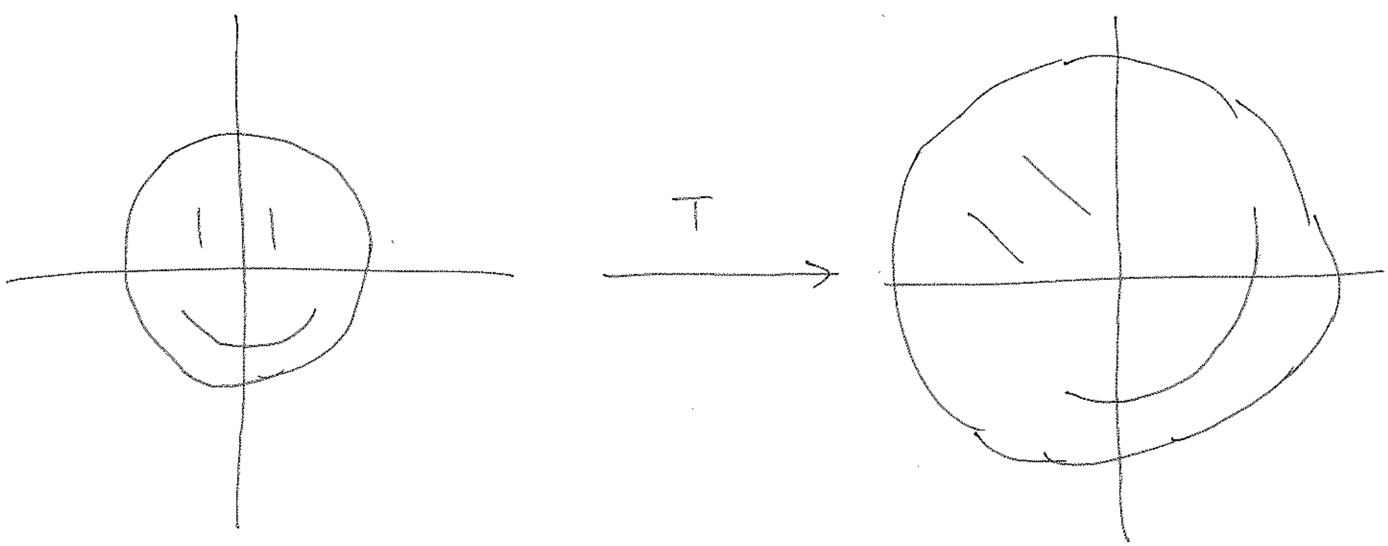
Quiz 6.



$$T\left(\begin{bmatrix}1\\-2\end{bmatrix}\right) = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}1+2\\1-2\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix}$$

$$T\left(\begin{bmatrix}-1\\-2\end{bmatrix}\right) = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}-1\\-2\end{bmatrix} = \begin{bmatrix}-1+2\\-1-2\end{bmatrix} = \begin{bmatrix}1\\-3\end{bmatrix}$$



T rotates everything to the left by 45° and expands it by a factor of $\sqrt{2}$.

Quiz 6 cont.

The nullspace $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

Row reduce $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow[\text{from R2}]{\text{Sub R1}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \end{array}\right]$

$\xrightarrow{\text{Mul R2 by } \frac{1}{2}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right]$

$\xrightarrow{\text{Add R2 to R1}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

$x = y = 0$, the nullspace is $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

The image is all of $\mathbb{R}^2$ because $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are in it, and these vectors are nonzero and not parallel.

Explain why:

* The image of a linear transformation T is equal to the span of the set of columns of the associated matrix.

Recall that the columns are $T\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right), T\left(\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}\right), \cdots T\left(\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}\right)$

and that

$$T\left(\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}\right) = x_1 T\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right) + x_2 T\left(\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}\right) + \cdots + x_n T\left(\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}\right)$$

The image of a linear transformation is, by definition,

$$\left\{ T\left(\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}\right) : x_1, \dots, x_n \in \mathbb{R} \right\}$$

$$= \left\{ x_1 T\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right) + \cdots + x_n T\left(\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}\right) : x_1 \dots x_n \in \mathbb{R} \right\}$$

(by above)

$$= \operatorname{Span}\left(\left\{ T\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right), \dots, T\left(\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}\right) \right\} \right)$$

by definition of "span", as desired.

* The nullspace contains only the zero vector if and only if the columns are linearly independent.

By the above, the nullspace contains a nonzero $\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$ if and only if we have

$$x_1 T\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right) + \cdots + x_n T\left(\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}\right) = \vec{0}.$$

This equation has a nontrivial solution precisely when the columns $T\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right), \dots, T\left(\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}\right)$ are linearly dependent.

(This was one of our criteria for linear dependence.)