Homework 7 - Math 544, Frank Thorne (thorne@math.sc.edu)

Monday, November 2, 2015

Each evening, a USC student chooses to either go out or stay home. Assume that this particular student's choices are modeled by a Markov chain with transition matrix

$$M = \begin{bmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{bmatrix},$$

where each evening's choice is based on the previous evening's choice, and the first row and column correspond to going out.

(i) Describe in a couple of sentences what the transition matrix is telling you about the student's preferences. (4)

(ii) Compute M^2, and again briefly describe what it is telling you. (4)
(4)

(iii) Compute a steady state vector for the Markov chain.
(8)

$$M = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \text{Out} & \text{Home} \\ \left[\begin{array}{cc} 0.1 & 0.4 \\ 0.9 & 0.6 \end{array}\right] \end{array}\begin{array}{l} \\ \text{Out} \\ \text{Home} \end{array}$$

Usually the student prefers to stay home. She is especially likely to stay home if she went out the previous night; she doesn't like to be out two nights in a row.

$$M^2 = \begin{bmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{bmatrix}\begin{bmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.1\cdot 0.1 + 0.4\cdot 0.9 & 0.1\cdot 0.4 + 0.4\cdot 0.6 \\ 0.9\cdot 0.1 + 0.6\cdot 0.9 & 0.9\cdot 0.4 + 0.6\cdot 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.37 & 0.28 \\ 0.63 & 0.72 \end{bmatrix}$$

This predicts her choice in two days, given her choice today. She is slightly more likely to go out if she goes out today, compared to if she stays home today. This is the opposite of the behavior in M.

$$\begin{bmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{bmatrix} -0.9 & 0.4 \\ 0.9 & -0.4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -0.9 & 0.4 & 0 \\ 0.9 & -0.4 & 0 \end{array}\right] \xrightarrow[\text{to R2}]{\text{Add R1}} \left[\begin{array}{cc|c} -0.9 & 0.4 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{Mul R1 by } -\frac{10}{9}} \left[\begin{array}{cc|c} 1 & -\frac{4}{9} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So x_2 is free subject to $x_1 - \frac{4}{9}x_2 = 0$, $x_1 + x_2 = 1$.
Since $x_1 = \frac{4}{9}x_2$, $\frac{13}{9}x_2 = 1 \Rightarrow x_2 = \frac{9}{13}$ and $x_1 = \frac{4}{13}$.

$\begin{bmatrix} 4/13 \\ 9/13 \end{bmatrix}$ is a steady state vector.

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Due Monday, November 2, 2015 SUBJECT TO ADDITIONS.

(1) Do A 1 (a-c), 12-14 from Knop.

(2) For each of the Markov chains below:

  (i) Describe in a couple of sentences what the transition matrix is telling you. You don't have to describe the matrix completely; just point out some of its most interesting features.

  (ii) Compute the square of the transition matrix and briefly describe it.

  (iii) Compute a steady state vector for the Markov chain.

  (iv) For *one* of the Markov chains (your choice) with transition matrix M, compute M^3, M^4, and M^5. Describe these as well. Do the values appear to be converging to the steady state vector?

  (a) This matrix predicts the outcome of a Gamecocks football win, based on the outcome of their previous game. The states are: Gamecocks win, Gamecocks lose.

$$\begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$$

  (b) Suppose you take one math class a semester. This matrix predicts your grade, based on the grade you earned in the previous course. The states are: A, B, C, D/F.

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 & 0.0 \\ 0.2 & 0.5 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0.0 & 0.1 & 0.1 & 0.6 \end{bmatrix}$$

  (c) This matrix predicts how well you sleep in a given night, based on how well you slept the previous night. The states are: good night of sleep, so-so, insomnia.

$$\begin{bmatrix} 0.7 & 0.2 & 0.5 \\ 0.2 & 0.5 & 0.2 \\ 0.1 & 0.3 & 0.3 \end{bmatrix}$$

  (d) This matrix predicts how likely you are to go to the gym and exercise, based on whether or not you exercised the previous day. The states are: exercised, did not exercise.

$$\begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$$

(e) This matrix predicts the day of the week. The states are Sun, M, T, W, Th, F, Sat.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(Hint: it is easier to find a steady state vector by thinking about it, rather than by a brute force computation.)

(f) Create your own! Describe a scenario which can be modeled by a Markov chain, write down a suitable transition matrix, and answer the same questions.

(g) **(Extra Credit 1)** The following Markov chain predicts whether or not today is Tuesday. The states are Tuesday and Not Tuesday.

$$\begin{bmatrix} 0 & \frac{1}{6} \\ 1 & \frac{5}{6} \end{bmatrix}$$

Writing M for the matrix, compute M^2. Explain why M is 'correct' but M^2 is 'wrong'. How can this be? Discuss.

(h) **(Extra Credit 2)** Here is a website that uses Markov chains to generate nonsense text:

`http://www.bitsofpancake.com/programming/markov-chain-text-generator/`

Here is a more elaborate website that does essentially the same thing, on a larger scale:

`https://pdos.csail.mit.edu/archive/scigen/`

The authors managed to get their nonsense accepted at computer science conferences, and they gave nonsense talks at them. You can generate your own!

Read these websites, experiment with them, and briefly summarize the basic idea of how they work.

1. (a) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot 5 + 2\cdot 0 & 1\cdot 1 + 2\cdot 1 & 1\cdot(-1) + 2\cdot 0 \\ 0\cdot 5 + 3\cdot 0 & 0\cdot 1 + 3\cdot 1 & 0\cdot(-1) + 3\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 0 \end{bmatrix}.$$

$\begin{bmatrix} 5 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ does not exist.

The number of columns in B does not equal the number of rows in ~~A~~ A.

HW (f) (Answers may vary!)