Quiz 8 - Math 544, Frank Thorne (thorne@math.sc.edu)

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Determine whether or not

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$. Explain your reasoning.

We want to determine if

$$a\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

can be solved for <u>any</u> x_1, x_2, x_3.

Row reduce:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 2 & 1 & 0 & x_2 \\ 0 & 1 & -1 & x_3 \end{array}\right] \xrightarrow[\text{from } R2]{\text{sub } 2R1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 0 & 1 & -2 & x_2 - 2x_1 \\ 0 & 1 & -1 & x_3 \end{array}\right]$$

sub R2 from R3

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 0 & 1 & -2 & x_2 - 2x_1 \\ 0 & 0 & 1 & x_3 - x_2 + 2x_1 \end{array}\right] \xrightarrow[\text{Add } 2R3 \text{ to } R2]{\text{sub } R3 \text{ from } R1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -x_1 + x_2 - x_3 \\ 0 & 1 & 0 & 2x_1 - x_2 + 2x_3 \\ 0 & 0 & 1 & 2x_1 - x_2 + x_3 \end{array}\right]$$

our matrix tells us how to choose a, b, c.
In particular there is a solution, so these three vectors <u>span</u> $\mathbb{R}^3$.
They are also <u>linearly independent</u>, because if $x_1 = x_2 = x_3 = 0$, then we see $a = b = c$ from the same matrix.

Therefore these three vectors form a basis for $\mathbb{R}^3$.

4.4 / B5.

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0 , y \geq 0 \right\} \subseteq \mathbb{R}^2 .$$

Show H spans $\mathbb{R}^2$.

Proof. Given $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, we want to write it as a linear combination of vectors in H. We may write

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix}$$

$$= c_1 \begin{bmatrix} |a| \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ |b| \end{bmatrix}$$

where $\begin{cases} c_1 = 1 \text{ if } a \geq 0 \text{ and } c_1 = -1 \text{ if } a < 0 \\ c_2 = 1 \text{ if } b \geq 0 \text{ and } c_2 = -1 \text{ if } b < 0 \end{cases}$

and both $\begin{bmatrix} |a| \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ |b| \end{bmatrix}$ are in H.

(Note: This solution is far from unique)

4.5 | A6,

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 2x - 3y = 0 \right\}.$$

A vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in the above has y free, but then $x = \frac{3}{2} r$.
(set $y = r$)

So we can write the above as

$$\left\{ \begin{bmatrix} 3/2\, r \\ r \end{bmatrix} : r \in \mathbb{R} \right\} = \left\{ r \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} : r \in \mathbb{R} \right\}$$

$$= \mathrm{Span}\left(\left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\} \right).$$

Since $\left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}$ is linearly independent (any set consisting of a single nonzero vector is), it is a basis.