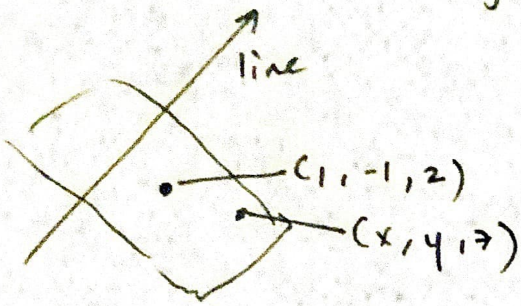
1. Find an equation for the plane perpendicular to
$x = 3t - 5$, $y = 7 - 2t$, $z = 8 - t$
that goes through $(1, -1, 2)$.



Ans. Our line is

$$(-5, 7, 8) + t(3, -2, -1)$$

and so we must have

$$(x, y, z) - (1, -1, 2)$$

perpendicular to $(3, -2, -1)$.

$$(x-1, y+1, z-2) \cdot (3, -2, -1) = 0$$

$$(3x - 3) + (-2)(y+1) - (z-2) = 0$$

$$3x - 3 - 2y - 2 - z + 2 = 0$$

$$\boxed{3x - 2y - z = 3.}$$

2. We say that a linear function $Df(\vec{a}) : \mathbb{R}^n \to \mathbb{R}^m$ is the derivative of f at $\vec{a}$ if the limit

$$\lim_{\|\vec{h}\| \to 0} \frac{f(\vec{a} + \vec{h}) - (f(\vec{a}) + Df(\vec{a})\vec{h})}{\|\vec{h}\|}$$

exists and equals zero. f is differentiable at $\vec{a}$ if it has a derivative $Df(\vec{a})$ as defined above.

The derivative measures the best linear approximation to $f(\vec{a} + \vec{h}) - f(\vec{a})$ when $\vec{h}$ is small. The function $f(\vec{a}) + Df(\vec{a})\vec{h}$ describes the generalization of the "tangent line" approximation to f when $\vec{h}$ is small.

[Optional but not actually required:]
If $f = (f_1, \dots, f_m)$ then Df is given by the ~~equation~~ matrix

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Can use the chain rule or do it directly.

Direct: $\vec{g}(r,\theta) = (r^2\cos^2\theta,\ r^2\sin\theta\cos\theta,\ r^2\sin^2\theta)$

$$D\vec{g} = \begin{bmatrix} \frac{\partial g_1}{\partial r} & \frac{\partial g_1}{\partial \theta} \\ \frac{\partial g_2}{\partial r} & \frac{\partial g_2}{\partial \theta} \\ \frac{\partial g_3}{\partial r} & \frac{\partial g_3}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 2r\cos^2\theta & -2r^2\cos\theta\sin\theta \\ 2r\sin\theta\cos\theta & r^2(\cos^2\theta - \sin^2\theta) \\ 2r\sin^2\theta & 2r^2\sin\theta\cos\theta \end{bmatrix}$$

Using chain rule.

$$D\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 0 \\ y & x \\ 0 & 2y \end{bmatrix}$$

Let's say $\vec{h}(r,\theta) = (x,y) = (r\cos\theta, r\sin\theta)$,

then $D\vec{h} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$.

Now $D\vec{f}$ as a function of r and θ is

$$\begin{bmatrix} 2r\cos\theta & 0 \\ r\sin\theta & r\cos\theta \\ 0 & 2r\sin\theta \end{bmatrix}$$

and $D\vec{g} = D\vec{f} \cdot D\vec{h}$

$$= \begin{bmatrix} 2r\cos\theta & 0 \\ r\sin\theta & r\cos\theta \\ 0 & 2r\sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

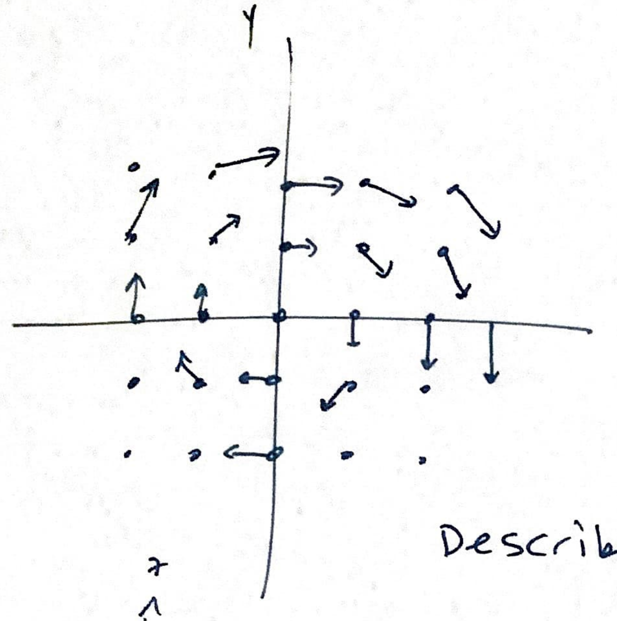
$$= \begin{bmatrix} 2r\cos^2\theta & -2r^2\cos\theta\sin\theta \\ 2r\sin\theta\cos\theta & -r^2\sin^2\theta + r^2\cos\theta \\ 2r\sin^2\theta & 2r^2\sin\theta\cos\theta \end{bmatrix}$$

Same as above!

(This way is probably harder.)

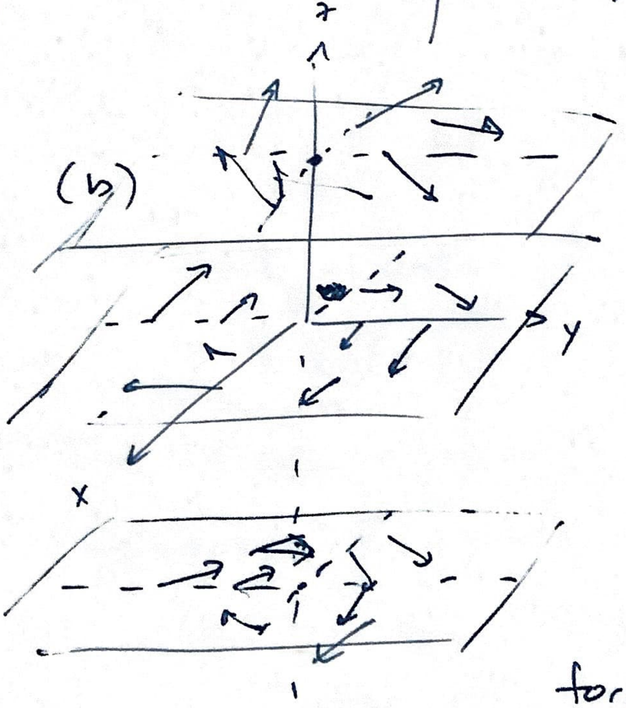
4. $\vec{F} = 2y\vec{i} - 2x\vec{j} + z\vec{k}$.

Picture of $2y\vec{i} - 2x\vec{j}$:



Describes clockwise rotation.

(b)



This is basically impossible to draw a good picture of. In ~~the~~ each x-y plane parallel to the xy-axes, the vector field describes forces acting in a clockwise rotation ~~else~~, but also upward or downward motion.

The motion is upward when $z > 0$ and down when $z < 0$, and is the same in each plane.

4c.

$$\mathrm{Div}\,\vec F = \nabla\cdot\vec F = \frac{\partial}{\partial x}(2y) + \frac{\partial}{\partial y}(-2x) + \frac{\partial}{\partial z}(z)$$

$$= 0 + 0 + 1 = 1.$$

$$\mathrm{Curl}\,\vec F = \vec\nabla\times\vec F = \det\begin{vmatrix} \vec i & \vec j & \vec k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & -2x & z \end{vmatrix}$$

$$\cancel{= \vec i\,(-2 - 2) +}$$

$$= \vec i\left(\frac{\partial}{\partial y} z - \frac{\partial}{\partial z}(-2x)\right) + \vec j\left(\frac{\partial}{\partial z} 2y - \frac{\partial}{\partial x} z\right)$$
$$+ \vec k\left(\frac{\partial}{\partial x}(-2x) - \frac{\partial}{\partial y}(2y)\right)$$

$$= \vec i\,(0-0) + \vec j\,(0-0) + \vec k\,(-2-2) = -4\vec k.$$

4d. Want a path $\vec x(t)$ such that

$$\vec x\,'(t) = \vec F(\vec x(t)).$$

~~The cheap~~ Try solution: circles!

Choose $\vec x(t) = (r\cos(-t), r\sin(-t), 0)$
$= (r\cos t, -r\sin t, 0)$.

(Choose $-t$ instead of t to make them go clockwise.)

Then $\vec x\,'(t) = (-r\sin t, -r\cos t, 0)$

and $\vec F(\vec x(t)) =$ ~~[illegible]~~ $(-2r\sin t,$ ~~[illegible]~~ $-2r\cos t, 0)$.

Oops. not the same.

4d (cont).

Since $\vec{x}'(t)$ is too small by a factor of two, speed it up.

Try $\vec{x}(t) = (r\cos(2t), -r\sin(2t), 0)$

$\vec{x}'(t) = (-2r\cos(2t), -2r\cos(2t), 0)$

$\vec{F}(\vec{x}(t)) = (-2r\sin(2t), -2r\cos(2t), 0)$

and now it works.

Get a solution for <u>every</u> $r > 0$, so more than two!
(Sorry, you don't get infinite extra credit.)

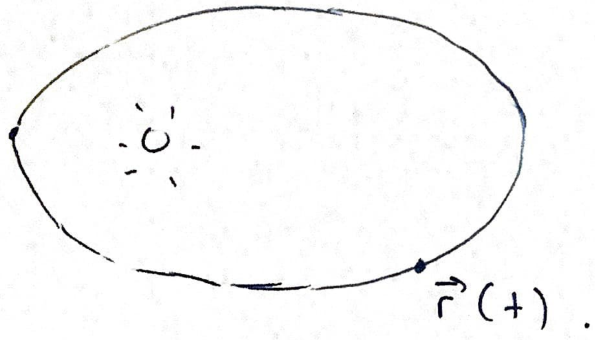
<u>Or</u>, ignore the maelstrom:

$\vec{x}(t) = (0, 0, e^t)$ works.

$\vec{x}'(t) = \vec{F}(\vec{x}(t)) = (0, 0, e^t)$.

$\vec{x}(t) = (0, 0, -e^t)$ is still another solution.

5.



(a) Let $\vec{v}(t) = \vec{r}\,'(t)$.

In this context $\dfrac{ds}{dt}$ is the speed of the planet (the pace at which it traverses is path).

We have

$$\vec{v}(t) = \frac{ds}{dt}\vec{T}$$

because $\vec{T}$ is the unit vector describing the direction of motion.

(b) Take the derivative of the above

$$\vec{a}(t) = \vec{v}\,'(t) = \frac{d}{dt}\left(\frac{ds}{dt}\vec{T}\right)$$

Note: These dots are ordinary multiplication of scalars.

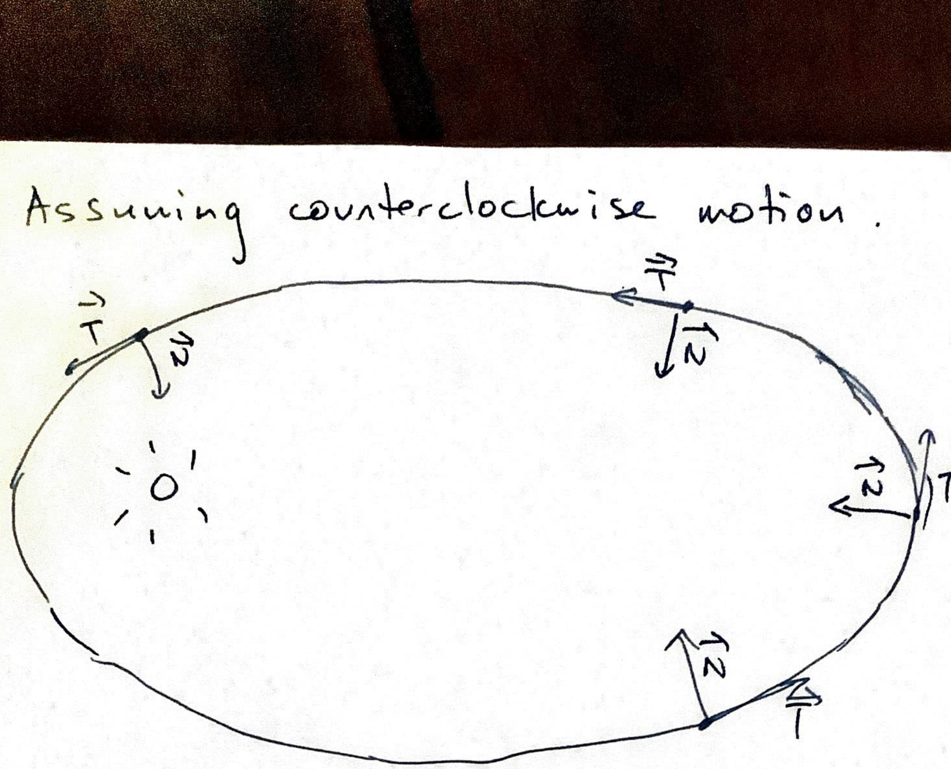
$$= \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\cdot\frac{d\vec{T}}{dt}$$

$$= \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\cdot\frac{d\vec{T}}{ds}\cdot\frac{ds}{dt}$$

(by the chain rule)

$$= \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2\cdot\kappa\vec{N}.$$

5(c). Assuming counterclockwise motion.



$\vec{B} = \vec{T} \times \vec{N}$ is constant, it is the unit vector sticking out of the plane.

(d) [Accidental trick question — sorry!
The hint does not make sense.]

As explained in (a), $\vec{v}(t) = \frac{ds}{dt}\vec{T}$, and $\vec{v}$ and $\vec{T}$ point in the same direction. $\frac{ds}{dt}$ is <u>always</u> positive as long as the planet is moving, which it always is.

(e) $\vec{a}(t) = \frac{d^2s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2\vec{N}$,

so $\frac{d^2s}{dt^2}$ is positive if and only if $\vec{a}(t)$ has a positive $\vec{T}$ component. Or, equivalently, when it is speeding up.
It will speed up when it moves towards the planet (here the hint in (a) is helpful; the distance to the sun is shrinking so it must speed up to sweep out equal areas) and slow down as it moves away.