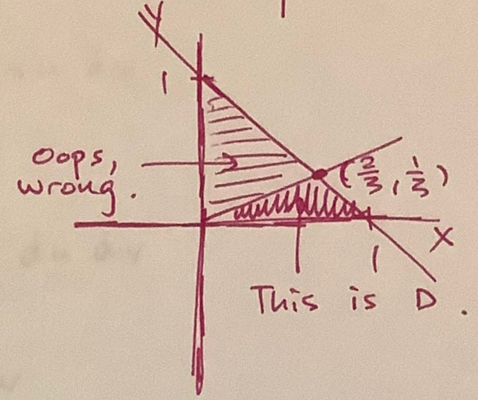
1. Compute $\iint_D \sqrt{\frac{x+y}{x-2y}}\, dA$,

where $D \subseteq \mathbb{R}^2$ is enclosed by $y = \frac{x}{2}$, $y = 0$, $x + y = 1$.



This is D.

Change of variables:

$u = x + y$

$v = x - 2y$

So: $u - v = 3y$

$y = \frac{u - v}{3}$

$2u = 2x + 2y$

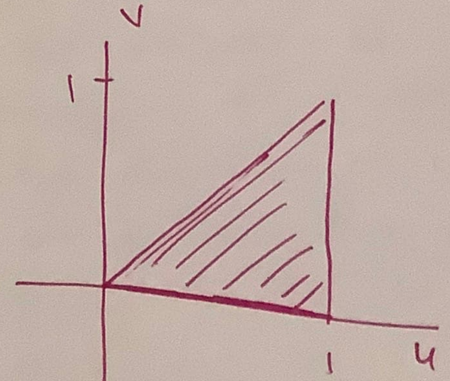
$v = x - 2y$

$2u + v = 3x \Rightarrow x = \frac{2u + v}{3}$.

$y = \frac{x}{2}$: $2 \frac{u - v}{3} = \frac{2u + v}{3}$

$\frac{2u - 2v}{3} = \frac{2u + v}{3}$

$\boxed{v = 0}$



$y = 0$: $\boxed{v = u}$

$x + y = 1$: $\frac{2u + v}{3} + \frac{u - v}{3} = 1$

$u = 1$.

1 (cont).

Our integral is

$$\iint_D \sqrt{\frac{x+y}{x-2y}}\, dA = \iint_{u,v} \sqrt{\frac{u}{v}}\, \left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

$$= \iint_{u,v} \sqrt{\frac{u}{v}}\, \left|\det \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right| du\, dv$$

$$= \iint_{u,v} \sqrt{\frac{u}{v}}\, \left|\det \begin{matrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{matrix}\right| du\, dv$$

$$= \frac{1}{3} \iint_{u,v} \sqrt{\frac{u}{v}}\; du\, dv$$

$$= \frac{1}{3} \int_{u=0}^{1} \int_{v=0}^{u} \sqrt{\frac{u}{v}}\; du\, dv$$

$$= \frac{1}{3} \int_{u=0}^{1} 2\sqrt{uv}\,\Big]_{v=0}^{u} du$$

$$= \frac{1}{3} \int_{u=0}^{1} 2u\, du = \frac{1}{3}\, u^2 \Big]_0^1 = \frac{1}{3}.$$

2. Let

$$C = \{ (x,y) \in \mathbb{R}^2 : f(x,y) = c \}$$

be a simple closed loop.

(a) Prove that $\int_C \nabla f \cdot d\vec{s} = 0$.

i.e. that $\int_C \left(\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} \right) \cdot d\vec{s} = 0.$

By Green's theorem it equals

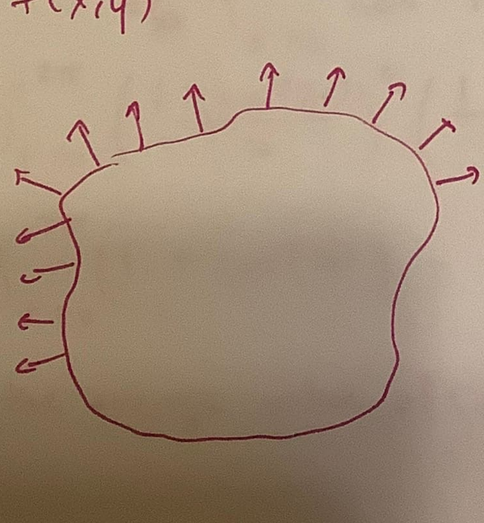
$$\iint_{\text{Interior of } C} \left(\frac{\partial}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial}{\partial y} \frac{\partial f}{\partial x} \right) dx\, dy$$

$$= \iint_{\text{Interior of } C} 0 \, dx\, dy \quad \text{(by equality of mixed partials)}$$

$$= 0.$$

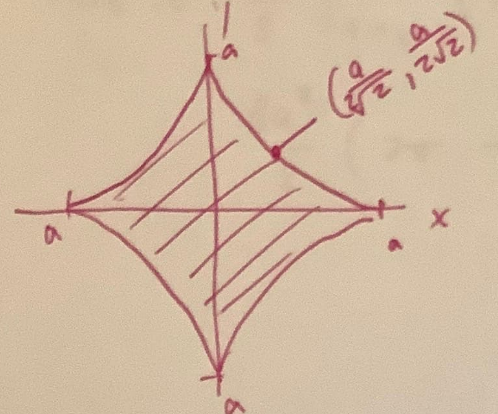
(This is not the only proof.)

(b) $f(x,y) = c$



Graphed is a level set of f. The gradient ∇f always points in the direction of fastest increase. $d\vec{s}$ always points in the direction of the curve, which is orthogonal to the gradient. So $\nabla f \cdot d\vec{s}$ is always zero. So its integral must be too.

(3) This is like a "scrunched circle"



By Green's theorem the area is

$$\oint_C x\, dy$$

$$= \int_{t=0}^{2\pi} a \cos^3 t \cdot a \cdot 3 \sin^2 t \cos t \, dt$$

$$= 3a^2 \int_{t=0}^{2\pi} \cos^4 t \sin^2 t \, dt$$

$$= 3a^2 \int_{t=0}^{2\pi} \left(\frac{1+\cos(2t)}{2}\right)^2 \left(\frac{1-\cos(2t)}{2}\right) dt$$

$$= \frac{3a^2}{8} \int_{t=0}^{2\pi} \left(1 + \cos(2t) - \cos^2(2t) - \cos^3(2t)\right) dt.$$

Since the average value of $\begin{cases} \cos(2t) = 0 & \text{over } [0, 2\pi] \\ \cos^3(2t) = 0 & \text{over } [0, 2\pi] \end{cases}$

$\uparrow$
This is the average x-coordinate in our picture

is zero, ~~and~~

3 (cont.)

This is $\frac{3a^2}{8} \int_{t=0}^{2\pi} \underbrace{\left(1-\cos^2(2t)\right)}_{\text{average value is } \frac{1}{2}} dt$

$= \frac{3a^2}{8} \left(2\pi - \pi\right) = \frac{3\pi a^2}{8}.$

4.

$$\vec{F} = \frac{x + xy^2}{y^2}\,\vec{i} - \frac{x^2+1}{y^3}\,\vec{j}.$$

Is $\vec{F} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$ for some f?

If so, then $\vec{F}$ is conservative and f is a scalar potential.

If $\frac{\partial f}{\partial x} = \frac{x + xy^2}{y^2}$

then $f = \dfrac{\frac{x^2}{2} + \frac{x^2}{2}y^2}{y^2} + g(y)$

$$= \frac{x^2 + x^2y^2}{2y^2} + g(y)$$

for a function $g(y)$ depending only on y.

If $\frac{\partial f}{\partial y} = -\frac{x^2+1}{y^3}$

then $f = \frac{x^2+1}{2y^2} + h(x)$

for a function $h(x)$ depending only on x.

4 cont.

So $f = \dfrac{x^2}{2y^2} + \dfrac{x^2}{2} + g(y)$

$f = \dfrac{x^2}{2y^2} + \dfrac{1}{2y^2} + h(x)$

The choice $f = \dfrac{x^2}{2y^2} + \dfrac{x^2}{2} + \dfrac{1}{2y^2}$ works!

And so $\vec{F}$ is conservative.

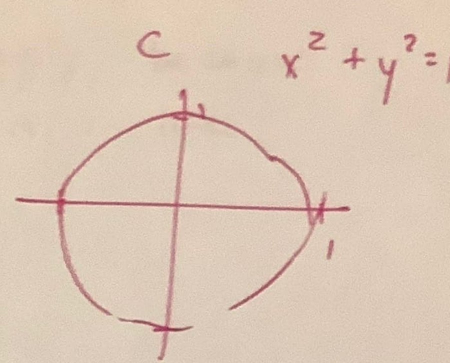
(iii) The work done by $\vec{F}$ in moving the particle from $(0,1)$ to $(1,1)$ is

$f(1,1) - f(0,1)$

$= \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) - \left(0 + 0 + \frac{1}{2}\right) = 1.$

The path doesn't matter in this case.

5. $\vec{F} = \dfrac{y}{x^2+y^2}\,\vec{i}$



Evaluate $\oint \vec{F}\cdot d\vec{s}$

Parametrize the curve:

$[0, 2\pi] \longrightarrow C$

$t \longrightarrow (\cos t, \sin t)$

Then $\vec{F}(t) = \dfrac{\sin t}{\cos^2 t + \sin^2 t}\,\vec{i} = \sin t\,\vec{i}$

$d\vec{s} = (-\sin t, \cos t)\,dt$

$\vec{F}(t)\cdot d\vec{s}(t) = \sin t \cdot (-\sin t)\,dt$

$= -\sin^2 t\,dt$

$\oint \vec{F}\cdot d\vec{s} = \int_0^{2\pi} -\sin^2 t\,dt = -\pi.$

More details:

$\int_0^{2\pi} -\sin^2 t\,dt$

$= \int_0^{2\pi} \dfrac{-1+\sin(2t)}{2}\,dt$

$= \left[\dfrac{-t}{2} - \dfrac{\cos(4t)}{2}\right]_0^{2\pi}$

$= \left[-\pi - \dfrac{1}{2}\right] - \left[0 - \dfrac{1}{2}\right] = -\pi.$

5(b).

No, Green's theorem does not apply, because $\vec{F}$ is not defined at $(0,0)$ which is in the interior of C.