Midterm Examination 2 - Math 141, Frank Thorne (thorne@math.sc.edu)

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Please work without books, notes, calculators, or any assistance from others.

1. (13 points) If a is any odd integer and b is any even integer, prove that $2a + 3b$ is even. (For this problem, use only the definitions of even and odd, and do not appeal to any previously established properties of even and odd integers.)

   Answer 1: You know that a is odd, and therefore $a = 2r + 1$ for some integer r. You know that b is even, and therefore $b = 2s$ for some integer s. Therefore,

   $$2a + 3b = 2(2r + 1) + 3(2s) = 4r + 2 + 6s = 2(2r + 3s + 1).$$

   We know that $2r + 3s + 1$ is an integer, so that $2a + 3b$ is twice an integer, and therefore is even.

   Answer 2: (This incorporates a small shortcut that you may have noticed.) You know that b is even, and therefore $b = 2s$ for some integer s. Therefore,

   $$2a + 3b = 2a + 3(2s) = 2(a + 3s).$$

   We know that $a + 3s$ is an integer, so that $2a + 3b$ is twice an integer, and therefore is even.

2. (13 points) Suppose that the product of three positive real numbers x, y, and z is at least 70. Prove that at least one of x, y, and z is greater than 4.

   We argue by contradiction. Suppose that x, y, and z are all positive integers which are less than or equal to 4. Then,

   $$x \cdot y \cdot z \leq 4 \cdot 4 \cdot 4 = 64,$$

   so that $xyz < 64$. However, this contradicts the assumption that $xyz \geq 70$. Therefore, at least one of x, y, and z is greater than 4.

3. (13 points) Determine whether the following statement is true or false, and prove or disprove it:

   If an integer a is of the form $5n + 1$ for some integer n, then a^2 is of the form $25m + 1$ for some integer m.

   False. We exhibit a counterexample. Let $n = 1$ so that $a = 6$. Then, $a^2 = 36 = 25 + 11$. By the unique division-with-remainder theorem, a^2 cannot be of the form $25m + 1$ if it is of the form $25b + 11$ (where $b = 1$).

4. (14 points) Prove that $\sqrt[3]{4}$ is irrational.

   You may use the following statement without proving it: For all integers a, if a^3 is even then a is even.

   Proof: Suppose to the contrary that $\sqrt[3]{4}$ is rational, so that we can write it as a fraction $\frac{a}{b}$, written where a and b are both positive and have no common factor. Then, cubing both sides of $\sqrt[3]{4} = \frac{a}{b}$, we get $4 = \frac{a^3}{b^3}$, so that $4b^3 = a^3$. Thus, a^3 is even, and so a is also even, and we can write $a = 2r$ for some integer r. We have $4b^3 = (2r)^3$, so that $b^3 = 2r^3$. Therefore, b^3 is even, and hence b is even also.

But this shows that a and b are both even and have the common factor 2, contrary to assumption. This is a contradiction; therefore, $\sqrt[3]{4}$ is irrational.

5. (14 points) Prove that $\lim_{x\to 3}(2x+1) = 7$.

   Proof: Suppose that $\epsilon > 0$ is given.

   [Aside: Not needed for proof, but shows you how to pick δ. If $2x+1 = 7+\epsilon$, then $x = 3+\epsilon/2$, and similarly if $2x+1 = 7-\epsilon$, then $x = 3-\epsilon/2$. So we should pick $\delta = \epsilon/2$, or anything smaller.]

   Choose $\delta = \epsilon/2$. Suppose that we are given x with $|x-3| < \delta$, i.e., $3-\epsilon/2 < x < 3+\epsilon/2$. Then, we have $2(3-\epsilon/2)+1 < 2x+1 < 2(3+\epsilon/2)+1$, i.e., $7-\epsilon < 2x+1 < 7+\epsilon$. In other words $|(2x+1)-7| < \epsilon$ whenever $|x-3| < \delta$. By definition, $\lim_{x\to 3}(2x+1) = 7$ as desired.

6. (14 points) Prove, for all integers $n \geq 1$, that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}.$$

   We prove this by induction. Let $P(n)$ be the claim that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}.$$

   Then $P(1)$ is true because both sides are equal to $1/2$. Suppose that $P(n)$ is true for some fixed integer n. We need to show that $P(n+1)$ is true. The left hand side of $P(n+1)$ is

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} + \frac{1}{(n+1)\cdot(n+2)}.$$

   By our inductive hypothesis (that $P(n)$ is true), this is equal to

$$\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}.$$

   This is equal to

$$\frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2},$$

   which is the right hand side of $P(n+1)$. Therefore $P(n+1)$ is true, and hence $P(n)$ is true for all $n \geq 1$ by induction.

7. (14 points) Prove that $1+3n \leq 4^n$ for every integer $n \geq 0$.

   We argue by induction. Let $P(n)$ be the claim $1+3n \leq 4^n$. Then $P(0)$ is true because both sides are equal to 1. Suppose now that $P(n)$ is true for some particular n. We want to prove that $P(n+1)$ is true.

   The left side of $P(n+1)$ is equal to $1+3(n+1) = (1+3n)+3$. By induction, this is less than $4^n + 3 \leq 4^n + 3\cdot 4^n = 4^{n+1}$, so that $P(n+1)$ is true. The result follows by induction.

8. (5 points) Let S be the set of integers divisible by 3, and let T be the set of integers divisible by 6. Do we have $S \subseteq T$? Do we have $T \subseteq S$?

   [For this problem, you do not have to give a proof or explanation (you should know how to – but time is short), but if your answer is wrong, this might be worth partial credit.]

   We have $T \subseteq S$ but not $S \subseteq T$. If x is an integer divisible by 6, then $x = 6r$ for some integer r, so that $x = 3(2r)$, so that x is a multiple of 3 (i.e., an element of S). To see that $S \not\subseteq T$, observe that 3 is in S but not T.