Selected Homework Solutions - Math 574, Frank Thorne

1. (4.5, 12). If a and b are rational numbers, $b \neq 0$, and r is an irrational number, then prove that $a + br$ is irrational.

   Proof. We argue by contradiction. Suppose that $a + br$ is rational. Then $br = (a + br) - a$ is rational, being a difference of two rational numbers. Also, $r = \frac{br}{b}$ is rational, being a quotient of two rational numbers (the denominator of which is not zero). However, we assumed that r was irrational, and r cannot be both rational and irrational.

   This is a contradiction, and therefore $a + br$ is irrational.

2. (4.5, 15). Prove that if a, b, and c are integers and $a^2 + b^2 = c^2$, then at least one of a and b is even.

   Proof: We argue by contradiction. Suppose that both a and b are odd. Then we can write $a = 2m + 1$ and $b = 2n + 1$ for integers m and n, and therefore

   $$a^2 + b^2 = (2m + 1)^2 + (2n + 1)^2 = 4m^2 + 4m + 1 + 4n^2 + 4n + 1 = 4(m^2 + m + n^2 + n) + 2.$$

   We divide into two cases: c is even, or c is odd. If c is odd, then so is c^2. However, the calculation above showed that $a^2 + b^2$ is even, and this is a contradiction. If c is even, then it is divisible by 2, and so c^2 is divisible by 4. However, $a^2 + b^2$ is equal to a multiple of 4 plus 2, and so it is not divisible by 4. In either case we have a contradiction. Therefore, at least one of a and b is even.

3. Prove that $\sqrt{2} + 2$ is irrational.

   Proof. We use the results, previously proved, that $\sqrt{2}$ is irrational, and that the sum of two rational numbers is rational.

   Suppose to the contrary that $\sqrt{2} + 2$ is rational. Then $\sqrt{2} = -2 + (\sqrt{2} + 2)$ is the sum of two rational numbers, hence rational. However, we know that it's irrational, and this is a contradiction. Therefore $\sqrt{2} + 2$ is irrational.

4. Prove that $\sqrt[3]{3}$ irrational.

   We have to argue the following claim first: Suppose that for some integer n, n^3 is divisible by 3. Then n is also divisible by 3.

   To prove this, we argue by contradiction. Suppose that n is not divisible by 3. In this case we can write $n = 3a + 1$ or $n = 3a + 2$ for some integer a, by the division-with-remainder theorem.

   If $n = 3a + 1$, then

   $$n^3 = (3a + 1)^3 = 27a^3 + 27a^2 + 9a + 1 = 3(9a^3 + 9a^2 + 3a) + 1,$$

   so that n^3 is of the form $3b + 1$, hence it is not divisible by 3.

   If $n = 3a + 2$, then

   $$n^3 = (3a + 2)^3 = 27a^3 + 54a^2 + 36a + 8 = 3(9a^3 + 18a^2 + 12a + 2) + 2,$$

   so that n^3 is of the form $3b + 1$, hence it is not divisible by 3.

   In either case, we get a contradiction. Therefore n is divisible by 3.

   Now we prove the main claim. Suppose to the contrary that we can write $\sqrt[3]{3}$ as a fraction $\frac{a}{b}$, where a and b have no common factor. Then, cubing, we have $3 = \frac{a^3}{b^3}$ and therefore $3b^3 = a^3$. Thus, a^3 is divisible by 3 and so a is also (by the claim argued above). Write $a = 3c$ for some integer c so that $3b^3 = 27c^3$, and thus $b^3 = 9c^3$. Therefore b^3 is divisible by 3, and hence b is also.

   But then a and b are both divisible by 3, contradicting the assumption that they have no common factor.

5. Prove that $\lim_{x \to 2} 0 = 0$. Suppose that any $\epsilon > 0$ is given. Then let $\delta = 37$. (Note: Any choice of δ whatsoever works.) Then, we must prove that whenever $|x - 2| < \delta$, we have $|0 - 0| < \epsilon$. However, the latter conclusion is true regardless of x, and so this holds. Therefore, $\lim_{x \to 2} 0 = 0$.

6. Prove that $\lim_{x \to 2} -2x - 9 = -13$.

   Suppose $\epsilon > 0$ is given.

[Aside: This is not needed in the proof, but this calculation tells you what δ to pick. If $-2x - 9 = -13 + \epsilon$, then $x = 2 - \epsilon/2$, and if $-2x - 9 = -13 - \epsilon$, then $x = 2 + \epsilon/2$. So we should choose $\delta = \epsilon/2$, or any smaller δ.]

Let $\delta = \epsilon/2$, and suppose that $|x - 2| < \delta$. Then this means that $2 - \delta < x < 2 + \delta$, so that $2 - \epsilon/2 < x < 2 + \epsilon/2$. Multiplying by -2 and subtracting 9, we obtain $-13 + \epsilon > -2x - 9 > -13 - \epsilon$, so that $|(-2x - 9) - (-13)| < \epsilon$ whenever $|x - 2| < \delta$, as required.

7. Prove that $\lim_{x \to \pi/4} \sin(x) \neq 1$. Let $\epsilon = 0.99 - \frac{\sqrt{3}}{2}$, and suppose that some $\delta > 0$ is given. We must prove that there exists x such that $|x - \pi/4| < \delta$ and $|\sin(x) - 1| > \epsilon$.

   We choose $x = min(\pi/3, \pi/4 + \delta/2)$. Then $|x - \pi/4| < \delta$ and x is between $\pi/4$ and $\pi/3$, so that $\sin(x)$ is between $\sqrt{2}/2$ and $\sqrt{3}/2$. This implies that $1 - \sin(x)$ is at least $1 - \sqrt{3}/2$, which is greater than $\epsilon = 0.99 - \frac{\sqrt{3}}{2}$. Therefore, $\lim_{x \to \pi/4} \sin(x) \neq 1$.

8. (5.3, 18). Prove that $5^n + 9 < 6^n$, for integers $n \geq 2$.

   Proof. Let $P(n)$ be the statement $5^n + 9 < 6^n$. Observe that $P(2)$ is true because $5^2 + 9 = 34 < 36 = 6^n$.

   Now, suppose that $P(n)$ is true for some n. Then, we have the inequalities

$$5^{n+1} + 9 = 5 \cdot 5^n + 9 < 5 \cdot (5^n + 9) < 5 \cdot 6^n < 6 \cdot 6^n = 6^{n+1}.$$

   Therefore, $P(n + 1)$ is true, and the result follows by induction.

   Comment: When writing induction proofs, please do not write down $P(n + 1)$ and then write a chain of statements leading to something you know is true. Although this can often be turned into a correct proof, this is backwards (it is the converse of what you are trying to do).

9. (6.2, 9) Prove that for any sets A, B, C, $(A - B) \cup (C - B) = (A \cup C) - B$.

   Proof. First, we prove that $(A-B) \cup (C-B) \subseteq (A \cup C) - B$. Suppose that $x \in (A-B) \cup (C-B)$. Then, either x is in $A - B$ or x is in $C - B$ (or both). If x is in $A - B$, then $x \in A$ and $x \notin B$. Since $x \in A$, we have $x \in (A \cup C)$, and therefore $x \in (A \cup C) - B$.

   If x is in $C - B$, then $x \in C$ and $x \notin B$. Since $x \in C$, we have $x \in (A \cup C)$, and therefore $x \in (A \cup C) - B$.

   Since $x \in (A \cup C) - B$ in either case, we can conclude that $(A - B) \cup (C - B) \subseteq (A \cup C) - B$.

   Now, we must prove that $(A \cup C) - B \subseteq (A - B) \cup (C - B)$. Suppose that $x \in (A \cup C) - B$. Then, $x \in A \cup C$, and $x \notin B$.

   Since $x \in A \cup C$, either $x \in A$ or $x \in C$ (or both). If $x \in A$, since $x \notin B$, we have $x \in A - B$. If $x \in C$, since $x \notin B$, we have $x \in C - B$. In either case, x is in at least one of $A - B$ and $C - B$, and therefore $(A \cup C) - B \subseteq (A - B) \cup (C - B)$.

   It thus follows that $(A \cup C) - B = (A - B) \cup (C - B)$.

   [Note: Sometimes I wrote 'is in' in English, and sometimes I used $\in$. These mean the same thing and are interchangeable.]