1.

(a) $15x + 21y = 51$ is equivalent to

$5x + 7y = 17$.

By inspection we see that $x = 2, y = 1$ is a solution.
Then all solutions are given by

$x = 2 - 7t$

$y = 1 + 5t$ for all integers t.

(b) We have $3 \mid 15x + 21y = 3(5x + 7y)$

and $3 \nmid 52$

so $15x + 21y = 52$ has no solutions.

(2)

(a) $11x \equiv 22 \pmod{31}$

Equivalent to

$x \equiv 2 \pmod{31}$ since $(11, 31) = 1$.

(b) $11x \equiv 22 \pmod{33}$.

Here when dividing by 11 we have to divide the modulus as well.

$x \equiv 2 \pmod{3}$.

(c) $11x \equiv 21 \pmod{33}$.

No solutions.

(Mod 11), the left side is 0

the right side is $21 \pmod{11} \equiv 10 \pmod{11}$

and $0 \not\equiv 10 \pmod{11}$.

(3) Least residue of 7^{2020} (mod 11).

We know $7^{10} \equiv 1 \pmod{11}$ by Fermat's Little Thm.

So $7^{2020} = (7^{10})^{202} \equiv 1^{202} \equiv 1 \pmod{11}$

(4) Solve $x \equiv 4 \pmod 9$ and $x \equiv 7 \pmod{11}$.

Since $(9, 11) = 1$ the answer is a unique congruence (mod 11).

The quick and dirty solution:

$x \equiv 4 \pmod 9$: 4, 13, 22, 31, 40, 49,

$x \equiv 7 \pmod{11}$: 7, 18, 29, 40, 51, ...

we see 40 in both lists so $\boxed{x \equiv 40 \pmod{99}}$

(5) Prove $(k, n+k) = d$ iff $(k, n) = d$.

Suppose that a is a common divisor of k and n + k.

Then $a \mid (n+k) - k = n$. So a is a common divisor of ~~k~~ k and n.

Conversely, suppose that b is a common divisor of k and n. Then $b \mid k + n$ too, so b is a common divisor of k and n + k.

So the set of common divisors of k and n + k is the same as the set of common divisors of k and n. So both sets have the same greatest element.

(6) Let p be an odd prime.

We know $(p-1)! \equiv -1 \pmod p$.

What is $(p-2)! \pmod p$?

Notice that $(p-1)! = (p-2)!(p-1)$.

$$\text{So } (p-2)!\,(p-1) \equiv -1 \pmod p$$

$$(p-2)!\,(-1) \equiv -1 \pmod p.$$

We cancel the -1 from ~~th~~ both sides. (This is okay since $(-1, p) = 1$.) We get

$$(p-2)! \equiv 1 \pmod p.$$

(7) Suppose $(a, m) = 1$.

Prove that $a, 2a, \dots (m-1)a$ represent different residue classes $\pmod m$.

Suppose to the contrary that $ra \equiv sa \pmod m$, where $1 \le r < s \le m-1$. Then we have $(s-r)a \equiv 0 \pmod m$ with $1 \le s-r \le m-2$. We have $m \mid a(s-r)$, and since $(m, a) = 1$ we have $m \mid s-r$. But this is impossible if $1 \le s-r < m$, a contradiction.

8.

$x \pmod{30}$

$x \pmod{24}$	0	1	2	3	...
0	0				
1		1			
2			2		
3					
$\vdots$					

(a) The box $1 \pmod{24}$, $0 \pmod{30}$, among many others, will be empty.

If $x \equiv 1 \pmod{24}$ then x is even and if $x \equiv 0 \pmod{30}$ then x is odd. No integer is both even and odd.

(b) What numbers between 0 and 719 appear in the same box as 0?

Those numbers which are divisible by both 24 and 30. We have

$24 = 2^3 \cdot 3$

$30 = 2 \cdot 3 \cdot 5$,

and by unique factorization, a number is divisible by both if and only if it is divisible by $2^3 \cdot 3 \cdot 5 = 120$.

So the multiples of 120:

0, 120, 240, 360, 480, 600.

(c) What numbers appear in the same box as 171 and 291?

Suppose $x \equiv 171 \pmod{24}$
$x \equiv 171 \pmod{30}$

Then $x - 171 \equiv 0 \pmod{24}$
$x - 171 \equiv 0 \pmod{30}$

So $x - 171$ has to be a multiple of 120 by (b),

(d) - (e). 120 boxes have 6 numbers
The rest have none.

~~A box~~
Given a box (a, b). Then this box contains a number iff $x \equiv a \pmod{24}$, $x \equiv b \pmod{30}$ is ~~poss~~ solvable. Since we have $x \equiv a \pmod 6$ and $x \equiv b \pmod 6$, we must have $a \equiv b \pmod 6$.

So only 120 ~~num~~ boxes $(\frac{1}{6} \cdot 720 = 120)$ can contain ~~boxe~~ numbers. Furthermore, by the argument in (c), if x and y are in the same box, then $x-y$ is a multiple of 120. So every box which contains a number contains exactly six of them.

Therefore, since there are $120 \cdot 6 = 720$ numbers, every box that can contain numbers does.