Homework 1, Math 701 – Frank Thorne (thorne@math.sc.edu)

Instructions: You are welcome and encouraged to collaborate, but please write up your own solutions.

Due Wednesday, September 9.

1. The *Fibonacci numbers* are defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Note that they satisfy

$$M \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

where $M := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, for each $n \geq 2$.

   (a) Compute the characteristic polynomial, eigenvalues, and a basis of eigenvectors for M.

   (b) Compute a diagonal matrix D and an invertible matrix P for which $D = P^{-1}MP$. Interpreting P as a change of basis matrix, describe the linear transformations and bases represented by D, P, and M.

   (c) Sketch a proof that the set of functions $f : \mathbb{Z}^+ \to \mathbb{R}$ satisfying $f(n+1) = f(n) + f(n-1)$ for $n \geq 2$ forms a two-dimensional real vector space V.

   (d) Using your work above, find a basis for V consisting of functions satisfying $f(n) = f(1)\lambda^{n-1}$ for some $\lambda \in \mathbb{R}$.

   (e) Write F in terms of this basis, thereby computing an explicit closed formula for the Fibonacci numbers.

2. Let V be the vector space of *affine plane conics*

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

with coefficients in $\mathbb{C}$. Then V is a vector space of dimension 6.

   (a) Fix (x_0, y_0), and let $W \subseteq V$ be the subspace of conics g for which $g(x_0, y_0) = 0$. Explain briefly why W is a subspace, and why it has dimension 5 for any choice of (x_0, y_0).

   (b) Now, for $k \geq 1$, consider the following statement: *For any k distinct points $(x_1, y_1), \ldots, (x_k, y_k)$, let $W_k \subseteq V$ be the subspace of conics g for which $g(x_i, y_i) = 0$ for each i. Then W_k has dimension $6 - i$.*

   For which $k \geq 1$ is the above statement true or false? (Prove your assertion.)

   (c) A conic $g(x, y)$ is *singular* if it contains some point (x_0, y_0) at which both partial derivatives vanish. Find (with proof) an example of a nonsingular conic, and a nonzero example of a singular conic.

   (d) Find a basis for V consisting of nonsingular conics.

3. Let V be the real vector space consisting of all continuous functions from $\mathbb{R}$ to itself. Prove that V is not finite dimensional.

4. (Lagrange Interpolation, following Friedberg, Insel, and Spence) Let V be the $n{+}1$-dimensional vector space of polynomials of degree $\leq n$, over a field F, and let $c_0, \ldots, c_n$ be distinct scalars in F.

   (a) For $0 \leq i \leq n$, define $f_i \in V^*$ by $f_i(p) = p(c_i)$. Prove that the f_i form a basis for V^*.

   (b) Prove that there exist unique polynomials $p_0, \ldots, p_n$ with $p_i(c_j) = \delta_{ij}$.

   (c) For any scalars $a_0, \ldots, a_n$ (not necessarily distinct), prove that there exists a unique polynomial q of degree $\leq n$, such that $q(c_i) = a_i$ for each i. Determine a formula for q in terms of the above.

5. (Suggested by ChatGPT) For each n, let P_n be the vector space of polynomials over $\mathbb{R}$ of degree at most n.

   (a) Let $D : P_3 \mapsto P_2$ be the differentiation map. Briefly explain why D is a homomorphism. Compute a matrix representing D (with respect to bases you should choose and specify).

   (b) Similarly to the previous question, for $a \in \mathbb{R}$ define ev_a to be the 'evaluation at a map', that is $\mathrm{ev}_a(p) = p(a)$. Explain why this is an element of P_n^*, and describe $D^*(\mathrm{ev}_a)$.

   (c) Define $I : P_n \mapsto \mathbb{R}$ by
   $$I(p) = \int_0^1 p(x)dx.$$
   Explain why $I \in P_n^*$.

   (d) Compute $D^*(I)$.