

Homework 3, Math 701 – Frank Thorne (thorne@math.sc.edu)

Instructions: You are welcome and encouraged to collaborate, but please write up your own solutions.

Due Monday, October 26, 2026.

1. Prove that a group of order 105 is not simple.
2. Prove that a group of order 132 is not simple.
3. Consider the group $\mathrm{SL}_2(\mathbb{Z}/3) = \mathrm{SL}_2(\mathbb{F}_3)$ of 2×2 matrices with entries in $\mathbb{Z}/3$ (i.e., in the finite field $\mathbb{F}_3$) whose determinant is 1.

(Note that the determinant involves multiplying as well as subtracting elements of $\mathbb{Z}/3$. If you are not familiar with finite fields, just note that both multiplication and addition are carried out modulo 3.)

- (a) Prove that this group has 24 elements in it.
- (b) Describe all of its 2-Sylow and 3-Sylow subgroups.
- (c) Prove that the center Z of $\mathrm{SL}_2(\mathbb{Z}/3)$ consists of $\pm I$.
- (d) Construct an action of $\mathrm{SL}_2(\mathbb{Z}/3)$ whose kernel is $\pm I$. This proves that $\mathrm{SL}_2(\mathbb{Z}/3)/Z$ is isomorphic to a subgroup of S_4 of order 12, and recall that the only such one is A_4 .

Hint: Try linear fractional transformations on $\{0, 1, -1, \infty\}$. Or, equivalently, $\mathrm{SL}_2(\mathbb{Z}/3)$ acts on $(\mathbb{Z}/3)^2$, in the usual way that matrices act on vectors; use this to somehow cook up an action on a set with four elements.

- (e) Prove that $\mathrm{SL}_2(\mathbb{Z}/3)$ is *not* isomorphic to S_4 .
4. Let G be the group of functions from $\mathbb{C}$ to itself of the form $z \mapsto az + b$, with $a \neq 0$.
 - (a) Prove that G is, in fact, a group, and compute a formula for the inverse of any element.
 - (b) Let A and B be the subgroups of maps $z \mapsto az$ and $z \mapsto z + b$, respectively. Prove that B is normal in G , and that A is not.
 - (c) Prove that G is isomorphic to a semidirect product $B \rtimes A$. Compute the associated homomorphism $A \mapsto \mathrm{Aut}(B)$ and the action of A on B by conjugation.
 5. Construct a non-abelian group of order 75.
 6. Classify all groups of order 28 (there are four isomorphism types).