Homework 4, Math 701 – Frank Thorne (thorne@math.sc.edu)

Instructions: You are welcome and encouraged to collaborate, but please write up your own solutions.

Due Friday, November 20, 2026.

1. Let F be a field. Prove that the matrix ring $M_n(F)$ is *simple*: it has no two-sided ideals other than 0 and itself.

2. Prove (with an explicit description) that $M_2(\mathbb{R})$ contains a subring isomorphic to $\mathbb{C}$.

3. Use the Chinese Remainder Theorem to prove the existence of a partial fraction decomposition: If $g(x) = \prod_{i=1}^{n}(x - a_i) \in \mathbb{C}[x]$ is a polynomial of degree n with distinct roots, and $f(x)$ is any polynomial of lesser degree, then we may write

$$\frac{f(x)}{g(x)} = \frac{b_1}{x - a_1} + \cdots + \frac{b_n}{x - a_n}$$

   for complex numbers b_i.

4. Prove:

   (a) The maximal ideals of $\mathbb{C}[x]$ are $(x - a)$ for $a \in \mathbb{C}$;

   (b) The only prime, nonmaximal ideal of $\mathbb{C}[x]$ is (0). (Recall that every maximal ideal is prime.)

5. Let R be the ring of continuous functions from $[0, 1]$ to $\mathbb{R}$.

   (a) Determine the units of R.

   (b) Determine all the zero divisors of R.

   (c) For each $0 \leq t \leq 1$, prove that $\{f \in R : f(t) = 0\}$ is a maximal ideal.

   (d) Prove that every maximal ideal is of this form.
   Hint: You will probably need a bit of real analysis in your proof.