26.1. Last time.

Suppose $\mathfrak{g}$ is a semisimple Lie algebra.
(no nonzero solvable ideals)
($\leftrightarrow$ every f.d. rep'n is semisimple: each invariant subspace has a complement)

Find $\mathfrak{h} \subseteq \mathfrak{g}$ an abelian subalgebra acting diagonally via the adjoint representation.

Then decompose $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha} \mathfrak{g}_\alpha \right)$
for roots $\alpha \in \mathfrak{h}^*$ s.t. $\mathrm{ad}(H)\, X = \alpha(H)\, X$ for all $H \in \mathfrak{h}$, $X \in \mathfrak{g}_\alpha$.

Now if V is some other irrep of $\mathfrak{g}$, then we can decompose $V = \bigoplus_{\beta} V_\beta$
and we have $\mathfrak{g}_{\beta} : V_\alpha \longrightarrow V_{\alpha+\beta}$.

So - find a highest weight vector
(according to some semi-orbitrary choice of positive roots)

- Push it around via elements in $\mathfrak{g}_\alpha$
- Get the entire irrep this way.
- So you can classify by their highest weight vectors.

[Recall the Killing form
$$B(X,Y) = \mathrm{Tr}\,(\mathrm{ad}\, X\ \mathrm{ad}\, Y) : \mathfrak{g} \to \mathfrak{g}$$
Review bottom of 25.6

26.4. (Note: 26.2 = 25.7, 26.5 = 25.8)

Let's up the ante.

* For a given semisimple $\mathfrak{g}$, classify all representations.

Let's classify semisimple Lie algebras while we're at it!

The root system determines the Lie algebra (probably this is not obvious) so we'll classify those.

Properties.

The roots R of $\mathfrak{g}$ span a real subspace of $\mathfrak{h}^*$ on which the Killing form is positive definite.

Call the space $\mathbb{E}$ and the Killing form $(-,-)$.

(1) R is finite and spans $\mathbb{E}$.

(2) $\alpha \in R \iff -\alpha \in R$, and $k\alpha \notin R$ for any $k \neq \pm 1$.

(3) For $\alpha \in R$, the reflection W_α in the hyperplane $\alpha^\perp$ maps R to itself.

(4) If $\alpha, \beta \in R$, the number
$$n_{\beta\alpha} = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$$
is an integer.

We saw (3), (4) before and (1) is true by construction.

Why is (2) true?

Consider the representation $\underline{i} = \bigoplus \mathfrak{g}_{k\alpha}$ of $\underline{sl_2} \cong \mathfrak{g}$.

Recall we could find subalgebras iso. to sl_2 acting on any chain of roots.

Let α be the smallest nonzero root in the string.

But can decompose $\underline{i} = \underline{sl_2} \oplus \underbrace{\underline{i}'}_{\text{no eigenvalue } 1 \text{ or } 2}$.

So must be trivial.

Property (4'). Let θ be the angle between α and β.

$$n_{\beta\alpha} = 2\cos(\theta)\frac{\|\beta\|}{\|\alpha\|} \in \mathbb{Z}$$

$$n_{\alpha\beta} = 2\cos(\theta)\frac{\|\alpha\|}{\|\beta\|} \in \mathbb{Z} \quad \Longrightarrow \text{ so } \quad \underline{4\cos(\theta)\in\mathbb{Z}.}$$

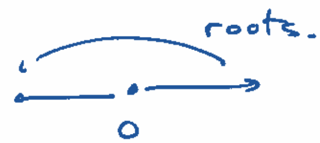
This means $\theta \in \left\{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}\right\}$.

The only angles whose cosines you actually know.

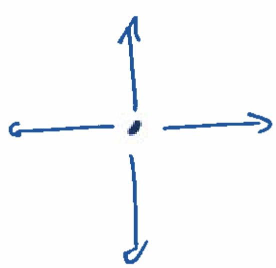
Write $n = \dim_{\mathbb{R}} E = \dim_{\mathbb{C}} \underline{h}$, the rank of the Lie algebra.

What can we get?

Rank 1. $(A_1, sl_2(\mathbb{C}))$



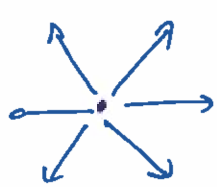
Rank 2. $(A_1 \times A_1)$



(not irreducible)

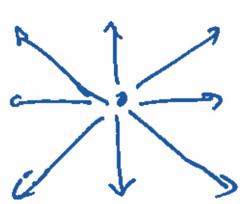
$sl_2(\mathbb{C}) \times sl_2(\mathbb{C}) \cong so_4(\mathbb{C})$.

(A_2)



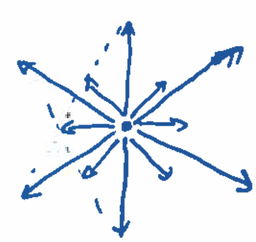
$sl_3(\mathbb{C})$

(B_2)



$so_5(\mathbb{C}) \cong sp_4(\mathbb{C})$

(G_2)



$\mathfrak{g}_2$

This is something that actually exists!

26.6.

Some more properties.

(5) If α, β are roots with $\beta \neq \pm\alpha$, then the α-string through β

$$\beta - p\alpha, \dots, \beta - \alpha, \beta, \beta + \alpha, \dots, \beta + q\alpha$$

has at most four in a row so $p + q \leq 3$. Also $p - q = n_{\beta\alpha}$.

Why? Use the Weyl group. Reflect across the α axis.

The reflection w_α flips the string, so

$$w_\alpha(\beta + q\alpha) = \cancel{w_\alpha(\beta - q\alpha)}\ \beta - p\alpha$$

but also $w_\alpha(\beta + q\alpha) = (\beta - n_{\beta\alpha}\alpha) - q\alpha$

so $p = q + n_{\beta\alpha}$.

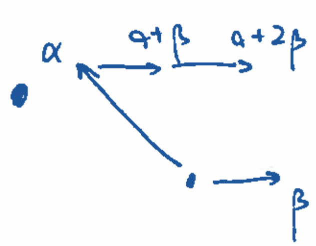
To get $p + q \leq 3$, take $p = 0$; $q = -n_{\beta\alpha} \leq 3$.

(6) If α, β roots with $\beta \neq \pm\alpha$, then:

$(\beta, \alpha) > 0 \implies \alpha - \beta$ is a root

$(\beta, \alpha) < 0 \implies \alpha + \beta$ is a root

$(\beta, \alpha) = 0 \implies \alpha - \beta, \alpha + \beta$ are both roots or both nonroots.



26.7

Call a root *simple* if it is positive and not the sum of two other positive roots.

(7) If α and β are distinct simple roots, then $\alpha - \beta$ and $\beta - \alpha$ are not roots.
(Immediate)

(8) The angle between two distinct simple roots cannot be acute. (Follows from (6), (7).)

(9) The simple roots are linearly independent.

> Exercise. If a set of vectors lies on one side of a hyperplane, and all angles are at least $90°$, the vectors are LI.

(10) There are precisely n simple roots.
(Immediate)

(11) Every positive root can be written uniquely as a nonnegative integral linear combination of simple roots.
(Uniqueness from 9)
(Existence: if it weren't, get another simple root).

To any root system associate a Dynkin diagram:

Circles $\longleftrightarrow$ simple roots.

Connect if the angle between them $> \frac{\theta}{2}$.

o —— o $\quad \frac{2\pi}{3}$ (roots will be of equal length)

o $\Rightarrow$ o $\quad \frac{3\pi}{4}$ (from longer to shorter)

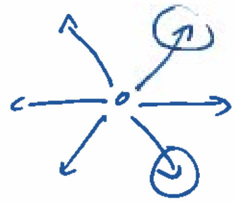
o $\Rrightarrow$ o $\quad \frac{5\pi}{6}$

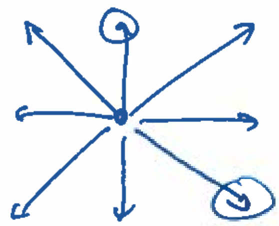
26.8.

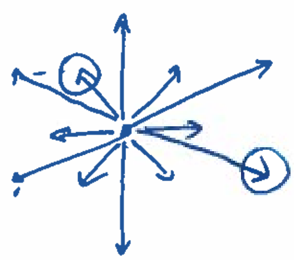
Examples.

Dynkin

A_1 : $sl_2(\mathbb{C})$ 

A_2 : $sl_3(\mathbb{C})$  

B_2 : $so_5 \cong sp_4$  

G_2 :  

Theorem: This is all possible Dynkin diagrams:

A_n $(sl_{n+1}(\mathbb{C}))$  $(n \geq 1)$

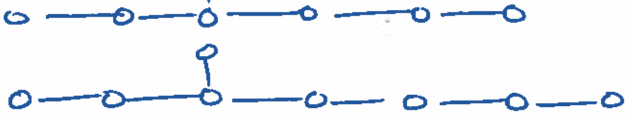
B_n $(so_{2n+1}(\mathbb{C}))$  $(n \geq 2)$

C_n $(sp_{2n}(\mathbb{C}))$  $(n \geq 3)$

D_n $(so_{2n}(\mathbb{C}))$  $(n \geq 4)$

E_6 

E_7 

E_8 

F_4 

G_2 

1.

Last Lie Groups class.

(1) Review what we've done.

(2) A loose end. How to reverse engineer the classification?

(3) A little bit about PHV's and my interests.

(4) Teaching evaluations.

Review:

Ch. 1 Introduced matrix Lie groups: closed subgps of $GL_n(\mathbb{C})$.

Examples. GL_n, SL_n, $U_n(\mathbb{C})$, $O_n(\mathbb{R})$ and $SO_n(\mathbb{R})$

And SU_n Also Sp_n.

Properties they might enjoy:

compactness

(path) connectedness

simple connectedness.

Proved $GL_n(\mathbb{C})$ is connected. Use Jordan form!

Lie groups in general: smooth manifolds G with a group structure;

$$G \times G \xrightarrow{\text{product}} G$$
$$G \xrightarrow{\text{inv.}} G$$

are smooth maps.

Ch. 2. The matrix exponential.

Analysis for matrices!

Defined $\exp: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ abs convergent

Recall $\|X\| = \left(\sum_{j,k} |x_{j,k}|^2\right)^{1/2} = (\operatorname{tr} X^* X)^{1/2}$

which satisfies $\|X + Y\| \leq \|X\| + \|Y\|$

$\|XY\| \leq \|X\| \cdot \|Y\|$.

Studied this analytically,

$$\frac{d}{dt} e^{tX} = X e^{tX} = e^{tX} X.$$

Remember that e^{X+Y} is <u>not</u> always $e^X e^Y$

It is <u>if</u> X and Y commute.

When X and Y don't commute, we have Baker-Campbell-Hausdorff.

One way to compute: If $X = CDC^{-1}$ then $e^X = C e^D C^{-1}$.

If D is <u>diagonal</u>, e^D is very easy to compute.
If D is upper triangular, it's not so bad.
So this can be used to compute e^X in practice.

There is also a matrix logarithm, again defined by power series.
But in general it only converges when $\|A - I\| < 1$.

So we have <u>locally</u> inverse bijections

$$\begin{array}{ccc} M_n(\mathbb{C}) & \underset{\log}{\overset{\exp}{\rightleftarrows}} & M_n(\mathbb{C}) \\ \uparrow & & \uparrow \\ 0 & & I \\ \hline \|X\| < \log 2 & & \|A - I\| < 1 \\ \Rightarrow \log e^X = X. & & \Rightarrow e^{\log A} = A. \end{array}$$

Lie Product Formula $\quad e^{X+Y} = \lim_{m \to \infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}} \right)^m$

Not exciting, but useful in proofs.

One parameter subgroups:

Any function $A : \mathbb{R} \longrightarrow GL_n(\mathbb{C})$ which is

(1) continuous, (2) $A(0) = I$, (3) a homomorphism (i.e. $A(t+s) = A(t)A(s)$)

Thm. All are of the form e^{tX} for some X.

Ch. 3. Lie algebras.

Introduced axiomatically (vector space with $[\cdot,\cdot]$ satisfying bilinearity, skew symmetry, Jacobi)

Usual example. Any associative algebra with $[X,Y] = XY - YX$.
(There's a whole classification theory.)
Defined homomorphisms, ad_X, irreducible (no ideals) simple (irred, dim ≥ 2) etc.

If G is a matrix Lie group, its Lie algebra is
$\mathfrak{g} = \{X : e^{tX} \in G \text{ for all } t \in \mathbb{R}\}$.
Check: You really do get a Lie algebra.
Also if G is commutative, so is $\mathfrak{g}$ (ie. bracket $\equiv 0$)

Computations:

$\mathfrak{gl}(n) = Mat_n$ (or just M_n)
$\mathfrak{sl}(n) = \{X \in M_n : tr(X) = 0\}$.
$\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) : X^* = -X\}$
$\mathfrak{o}(n) = \{X \in M_n(\mathbb{R}) : X^T = -X\}$.

Thm. Lie group homomorphisms induce Lie alg. homomorphisms.

Given $\Phi : G \to H$, there exists $\phi : \mathfrak{g} \to \mathfrak{h}$ defined by $$\phi(X) = \frac{d}{dt} \Phi(e^{tX}) \Big|_{t=0}$$

and satisfying $\Phi(e^X) = e^{\phi(X)}$.

i.e.
$$\begin{tikzcd} G \arrow[r, "\Phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\phi"] \arrow[ur, phantom, "\circlearrowleft"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

Defined a map $$\begin{aligned} Ad_A : \mathfrak{g} &\to \mathfrak{g} \\ X &\to AXA^{-1} \end{aligned}$$

(You do get an element of $\mathfrak{g}$.)

The map $A \to Ad_A$ is a Lie group homomorphism $G \to GL(\mathfrak{g})$, and its associated Lie algebra hom $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is ad ($X \to ad_X := Y \to [X,Y]$).

Thm 3.42. We have mutually inverse local homeomorphisms

$$\mathfrak{g} \underset{\log}{\overset{\exp}{\rightleftarrows}} G.$$

Did this for all matrices already. Point is that the images land in G and $\mathfrak{g}$ respectively.

One consequence. $\mathfrak{g}$ is the tangent space to the identity to G, i.e.

$$X \in \mathfrak{g} \longleftrightarrow \left\{ \exists \text{ smooth curve } \gamma : \mathbb{R} \to M_n(\mathbb{C}),\ \gamma(0) = I \text{ and } \left.\frac{d\gamma}{dt}\right|_{t=0} = X \right\}.$$

Another. If G is a connected matrix Lie group, every elt. $A \in G$ can be written

$$A = e^{X_1} e^{X_2} \cdots e^{X_m} \quad \text{for } X_1, \ldots, X_m \in \mathfrak{g}.$$

Another. If two Lie group homs induce the same Lie alg. hom, they're the same.

Ch. 4. Representation Theory.

Interested in (f.d., cpx) rep'ns

$$\Pi : G \to GL(V)$$
$$\pi : \mathfrak{g} \to \mathfrak{gl}(V)$$

(Recall Π induces π)

~~Given~~ Looked at various examples.

One: $SL(2)$ or $GL(2)$ or $SU(2)$ acting on $Sym^m(2)$

homo polys of degree m in 2 cpx vars.

Several ways to define (all isomorphic)

$$(\Pi_m(U) f)(z) = f(U^{-1} z) \quad \text{for } f \in Sym^m(2),\ U \in GL(2).$$

Usually you just see

$$U \circ f(z) = f(U^{-1} z).$$

6. We computed the associated Lie algebra rep'n. Saw the usual pattern.

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then under the Lie algebra rep'n, $z_1^{m-k} z_2^{k}$ is an eigenvector for H with eigenvalue $-m+2k$.

X and Y shift the exponent up/down by one.

* These rep'ns are all irreducible
* They are <u>all</u> the irred reps of $sl(2)$.
* Hence they describe the rep thy of $SL(2)$.

Ch. 5.

BCH and its consequences.

<u>BCH theorem</u>. If X, Y are in $M_n(\mathbb{C})$ with $\|X\|, \|Y\|$ small then

$$\log(e^X e^Y) = X + \int_0^1 g(e^{ad_X} e^{t\,ad_Y})(Y)\, dt$$

$$= X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]]$$

$$- \frac{1}{12}[Y, [X, Y]] + \cdots$$

The point is that it is described entirely in terms of the Lie bracket.

Consequences:

* Given $G, H, \mathfrak{g}, \mathfrak{h}$, $\phi : \mathfrak{g} \to \mathfrak{h}$.
  If G is simply connected, then ϕ determines a unique hom $\Phi : G \to H$ with $\Phi(e^X) = e^{\phi(X)}$ for $X \in \mathfrak{g}$.

  Cor. If G, H simply connected and $\mathfrak{g} \cong \mathfrak{h}$, then $G \cong H$.

  Construct a local isomorphism and extend.
  (In general: given G, H with G simply conn'd, a local homomorphism $G \to H$ extends uniquely to a global one.)

  If G is not simply connected you can consider its universal cover $\tilde{G}$ and get $\Phi : \tilde{G} \to H$.
  $\tilde{G}$ and G will have the same Lie algebra.

* Given $G, \mathfrak{g}, \mathfrak{h} \subseteq \mathfrak{g}$. There is a unique <u>connected</u> <u>Lie subgroup</u> H of G with Lie alg. $\mathfrak{h}$.
  (These aren't necessarily closed.)

* <u>Theorem</u>. If $\mathfrak{g}$ is any f.d. real Lie algebra, there is a connected Lie subgroup G of $GL(n, \mathbb{C})$ whose Lie alg. is iso to $\mathfrak{g}$.