11.1.

W. Rossmann, *Lie Groups* (basically the same)

Just working with matrices (no Lie groups or algebras yet)

$$Ad(A)\,Y = AYA^{-1}.$$

$$ad(X)\,Y = XY - YX.$$

Proposition. $Ad(\exp X) = \exp(ad\ X)$.

What does $\exp(ad\ X)$ mean?

It can only mean one thing: $1 + ad\ X + \frac{(ad\ X)^2}{2} + \frac{(ad\ X)^3}{6} + \cdots$

Both sides are operations on matrices, so the claim is, for all matrices Y of the same size,

$$Ad(\exp X)\,Y = \exp(ad\ X)\,Y$$

e.g.

$$(\exp X)\,Y\,(\exp X)^{-1} = Y + [X,Y] + \tfrac{1}{2}[X,[X,Y]] + \tfrac{1}{6}[X,[X,[X,Y]]] + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (ad\ X)^k\,Y.$$

Proof. Let $A(t) = Ad(\exp tX)$.

Then $A'(t)\,Y = \frac{d}{dt}\left(\exp(tX)\,Y\exp(-tX)\right)$

$= X\exp(tX)\,Y\exp(-tX) + \exp(tX)\,Y\exp(-tX)\cdot(-X)$

$= \mathrm{ad}(X)\ \mathrm{Ad}(\exp(tX))\,Y.$ } Stare at this for 15 seconds until you understand why it's a tautology!

$= \mathrm{ad}(X)\cdot A(t)\,Y$

Also $A(0) = I$ (as an operation on matrices, so an element of $GL(n^2)$ if we were starting with $n\times n$ matrices).

By what we have seen, the unique solution to this differential equation is $A(t) = \exp(t\ \mathrm{ad}\,X)$.

This (with $t=1$) is what we wanted!

Back to Hall, the machinery was set up as:

$G \xrightarrow{\Phi} H$ Lie group hom

$\mathfrak{g} \xrightarrow{\phi} \mathfrak{h}$ defined w/ $\Phi(e^X) = e^{\phi(X)}$

Indeed, get $\Phi(e^{tX}) = e^{tZ}$ simul. for all $t\in\mathbb{R}$ (classification of 1PS's)

11.3

We had $\mathrm{Ad}_A : \mathfrak{g} \longrightarrow \mathfrak{g}$
(or $\mathrm{Ad}(A)$) $X \rightarrow A X A^{-1}$

and $\mathrm{Ad}_A \, \mathrm{Ad}_B = \mathrm{Ad}_{AB}$ so this is a rep'n
$G \longrightarrow GL(\mathfrak{g})$.

also had $\mathrm{ad}_A : \mathfrak{g} \longrightarrow \mathfrak{g}$
$Y \rightarrow [A, Y] = AY - YA$.

This is a map $\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ (not $GL(\mathfrak{g})$)

Let us call by "ϕ" the Lie algebra map induced by $\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$. It will be a map $\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$.

How to define? Will have

$$\mathrm{Ad}(e^{tX}) = e^{tZ} \quad \text{for some } Z \in \mathfrak{gl}(\mathfrak{g})$$
(this is $\phi(X)$)

and (see again Thm 3.28) it can be computed by

$$\phi(X) = \frac{d}{dt} \mathrm{Ad}(e^{tX}) \Big|_{t=0}$$

$$= \frac{d}{dt} \left\{ Y \rightarrow e^{tX} Y e^{-tX} \right\} \Big|_{t=0}$$

$$= \left\{ Y \rightarrow \frac{d}{dt} e^{tX} Y e^{-tX} \Big|_{t=0} \right\}$$

$$= \left\{ Y \rightarrow \left(X e^{tX} Y e^{-tX} + e^{tX} Y e^{-tX} \cdot (-X) \right) \Big|_{t=0} \right\}$$

$$= \{ Y \rightarrow XY - YX \} = \mathrm{ad}(X).$$

which is what we damn well expected.

11.4

The exponential map.

Def. If G is a matrix Lie group, w/ Lie alg. $\mathfrak{g}$, the exponential map for G is the map

$$\exp : \mathfrak{g} \longrightarrow G.$$

We already know this is well-defined.

Thm. (true, but we didn't prove it) Every matrix in $GL_n(\mathbb{C})$ is e^X for some $X \in M_n(\mathbb{C})$.

But be careful.

Example. Let $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{C})$.

There does not exist $X \in \mathfrak{sl}_2(\mathbb{C})$ with $e^X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$.

Proof. If such X existed, if it were diagonalizable then A would be too.

So X has a repeated eigenvalue which must be 0 ($\mathfrak{sl}_2$: trace 0).

So $Xv = 0$ for some $v \Rightarrow e^X v = v$

So e^X has 1 as an EV.
But the above matrix does not.

So annoyances:

The exponential map may not be onto
It may not be one-to-one

but it is locally such.

Thm [3.42]. For each ~~Let~~ $0 < \varepsilon < \log 2$, ~~let~~ write

$$U_\varepsilon = \{ X \in M_n(\mathbb{C}) : \|X\| < \varepsilon \}$$
$$V_\varepsilon = \exp(U_\varepsilon)$$

Let $G \subseteq GL_n(\mathbb{C})$ matrix Lie group w/ Lie alg. $\mathfrak{g}$

Then: there exists $\varepsilon \in (0, \log 2)$ s.t. for $A \in V_\varepsilon$,

$$A \in G \longleftrightarrow \log A \in \mathfrak{g}.$$

What we've proved before: (Theorem 2.8)

We have a homeomorphism $U_\varepsilon \xrightarrow{\exp} V_\varepsilon$ for each such ε.

~~So~~ This lives within $\mathfrak{gl}(n) \to GL(n)$.

Proved it by defining log via power series, and proving this is a cts., convergent inverse.

What's new. True for any matrix Lie group G.

Don't have to work with all of $GL(n)$ to get this.

Lemma. Suppose $\{B_m\} \subseteq G$, a sequence converging to I.

Write $Y_m = \log B_m$. (Know defined for $m \gg 0$
WLOG defined for all m)

Suppose $Y_m \neq 0$ for all m, $\dfrac{Y_m}{\|Y_m\|} \longrightarrow Y \in M_n(\mathbb{C})$.

Then $Y \in \mathfrak{g}$.

Proof. For any $t \in \mathbb{R}$, choose integers k_m s.t. $k_m \|Y_m\| \to t$. Can do this because $\|Y_m\| \to 0$ and none of the Y_m are zero.

So $e^{k_m Y_m} = \exp\left(k_m \|Y_m\| \cdot \frac{Y_m}{\|Y_m\|} \right) \to e^{tY}$.

But $e^{k_m Y_m} = (e^{Y_m})^{k_m} = B_m^{k_m} \in G$.

Moral. $e^{tY} \in G$. So $Y \in \mathfrak{g}$. Done

Proof of theorem. Regard $M_n(\mathbb{C}) = \mathbb{R}^{2n^2}$

Write $D :=$ ortho complement of $\mathfrak{g}$.

Define a map $\Phi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$

$$Z \longrightarrow e^X e^Y$$

where we always write $Z = \underset{\substack{\uparrow \\ \mathfrak{g}}}{X} + \underset{\substack{\uparrow \\ D}}{Y}$.

This is cts. and $\frac{d}{dt} \Phi(tX, 0) \Big|_{t=0} = X$

$$\frac{d}{dt} \Phi(0, tY) \Big|_{t=0} = Y.$$

So the derivative of Φ at $0 \in \mathbb{R}^{2n^2}$ is the identity which is invertible.

11.7

<u>Inverse function theorem</u> $\Rightarrow$ Φ has a cts. local inverse (defined in a nbd of I)

<u>WTS</u>, if $A \in V_\varepsilon \cap G \Rightarrow \log A \in \mathfrak{g}$.

If not, there would be a sequence $A_m \in G \longrightarrow I$ with $\log A_m \notin \mathfrak{g}$ for all m.

Write $A_m = e^{X_m} e^{Y_m}$ $\quad (X_m \in \mathfrak{g},\ Y_m \in D)$ $\quad X_m \to 0 \quad Y_m \to 0.$

which we can do because Φ has a local inverse.

We have $Y_m \neq 0$ because $e^{\mathfrak{g}} \subseteq G$.

Since A_m and e^{X_m} are in G, so is e^{Y_m}.

<u>Now</u>, choose a subseq. of the Y_m (WLOG assume it's the whole sequence)

with $\dfrac{Y_m}{\|Y_m\|} \to Y \in D$. (Here $\|Y\| = 1$)

Why can we do this? $\dfrac{Y_m}{\|Y_m\|}$ is a unit vector

set of all such is <u>compact</u>

can find a subseq converging to <u>something</u>

and by construction it's in D.

By lemma, $Y \in \mathfrak{g}$. But $Y \in D = \mathfrak{g}^\perp$, <u>contradiction</u>. ∎

12.1 (Review the big theorem)

Cor. G matrix Lie gp w/ Lie alg. $\mathfrak{g}$.

There exist nbds $U \ni 0$ in $\mathfrak{g}$, $V \ni I$ in G s.t.

exp is a homeomorphism $U \to V$.

Proof. Given ε as in Thm, let $U = U_\varepsilon \cap \mathfrak{g}$

$V = V_\varepsilon \cap \mathfrak{g}$.

Then ~~onto~~ one-to-one and continuous by general theory.

Onto by theorem.

Cor. Let $k = \dim \mathfrak{g}$. (as an $\mathbb{R}$-vector space)

Then G is a smooth embedded submanifold of $M_n(\mathbb{C})$

of dimension k. (So it's a Lie group.)

Corollary of Corollary. Locally path connected ($\mathbb{R}^k$ is)

so path conn $\iff$ conn'd.

Proof involves a lot of topology terminology.

Idea. Have a homeomorphism $U \longrightarrow V$

choose an ε-box in U.

Get a nbd homeomorphic to $\mathbb{R}^k$ in U, hence in V.

But the map $V \longrightarrow gV$
$x \longrightarrow gx$ is a homeomorphism for every $g \in G$.

12.2. Cor. $(G, \mathfrak{g})$ as before.

Then $X \in \mathfrak{g} \iff \exists$ a smooth curve γ in $M_n(\mathbb{C})$,

$\gamma(t) \in G$ for all t,

$\gamma(0) = I$,

$$\left.\frac{d\gamma}{dt}\right|_{t=0} = X.$$

In other words: $\mathfrak{g}$ is the tangent space at the identity to G.

Proof. $\Rightarrow$: Choose $\gamma(t) = e^{tX}$.

$\Leftarrow$: Given such a γ, for small t write

$\gamma(t) = e^{\delta(t)}$ with δ a smooth curve in $\mathfrak{g}$.

Now

$$\gamma'(0) = \left.\frac{d}{dt} e^{\delta(t)}\right|_{t=0} \quad ~~= \frac{d}{dt} e^{t\delta'(0)}~~$$

$$= \left. e^{\delta(t)} \cdot \delta'(t)\right|_{t=0} = e^{0}\delta'(0) = \delta'(0).$$

And so $\delta'(0) = \gamma'(0)$ belongs to $\mathfrak{g}$.

Cor. If G is a conn'd matrix Lie group, every elt. of A can be written ~~as eX~~ in the form

$$A = e^{X_1} e^{X_2} \cdots e^{X_n}$$

for some $X_i \in \mathfrak{g}$.

Proof. (Not the book's, please check for ~~[crossed out]~~ mistakes!)

Let U be a symmetrical small nbd of $0 \in \mathfrak{g}$. Then e^U is open $\subset G$.

Consider everything that can be written

$$e^{X_1} \cdots e^{X_n} \quad \text{with all the } X_i \in U.$$

It's a subgroup of G.

H is open ("obviously")

All of its cosets are open, hence it's also closed

connected $\Rightarrow$ whole thing!

Cor. Given $G \overset{\Phi_1}{\underset{\Phi_2}{\rightrightarrows}} H$ two homs of Lie groups

with assoc. homs $\mathfrak{g} \overset{\phi_1}{\underset{\phi_2}{\rightrightarrows}} \mathfrak{h}$. If $\phi_1 = \phi_2$, $\Phi_1 = \Phi_2$.

Proof. $\Phi_1(e^{X_1} \cdots e^{X_m}) = e^{\phi_1(X_1)} \cdots e^{\phi_1(X_m)}$

$= \Phi_2(e^{X_1} \cdots e^{X_m})$

and this is everything.

Cor. If $G \xrightarrow{\Phi} H$ is a cts. homomorphism, it is smooth.

Proof. write ~~[crossed out]~~ a nbd. of arbitrary $g \in G$ as $\{ge^X : X \in \mathfrak{g}\}$.

Then if $h \in ge^X$, $\Phi(h) = \Phi(g)\,\Phi(e^X) = \Phi(g)\, e^{\phi(X)}$

So that Φ in this neighborhood is the composition of:

a linear map,

exponential map.

left mul by $\Phi(g)$. All are smooth

Cor. If G is a conn. matrix Lie group and $\mathfrak{g}$ is abelian, then so is G.

Proof. Write any $A \in G$ as $e^{X_1} \cdots e^{X_m}$ where all the X_i commute.

Cor. If $G \subseteq M_n(\mathbb{C})$ matrix Lie group, its identity component G_0 is a closed subgroup of $GL_n(\mathbb{C})$ and also a matrix Lie group.

It has the same Lie algebra as G.

Proof. Know it's a subgroup.

Open, and so are its cosets, so closed.

Lie algebras are the same, because the condition

$$\{e^{tX} \in G \text{ for all } t \in \mathbb{R}\}$$

if it is true, lies in G_0 since it is a smooth path.

13. 1

My favorite representation.

$V = \{\text{binary cubic forms}\}$

$= \{x_1 u^3 + x_2 u^2 v + x_3 u v^2 + x_4 v^3\}$.

(Can talk about $V(\mathbb{C})$, $V(\mathbb{R})$ etc.)

$G = GL(2)$, V is a rep'n by means of

$$(g \cdot f)(u,v) = f((u,v)g).$$

So writing $\rho : GL(2) \longrightarrow GL(4)$ for this rep'n, get

$$\rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^3 & a^2 b & ab^2 & b^3 \\ 3a^2 c & a^2 d + 2abc & 2abd + bc^2 & 3b^2 d \\ 3ac^2 & 2acd + bc^2 & ad^2 + 2bcd & 3bd^2 \\ c^3 & c^2 d & cd^2 & d^3 \end{pmatrix}$$

Is it faithful? If $\rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = I_4$, $b = c = 0$, looks like

$$\begin{pmatrix} a^3 & 0 & 0 & 0 \\ 0 & a^2 d & 0 & 0 \\ 0 & 0 & ad^2 & 0 \\ 0 & 0 & 0 & d^3 \end{pmatrix}$$

So $a = d \in \mu_3$.

(Interesting question. Compute isotropy subgroups.)

Now compute the infinitesimal rep'n dp.

What is this?

~~$\lim_{\alpha,\beta,\gamma,\delta \to 0} \begin{pmatrix} 1+\alpha & \beta \\ \gamma & 1+\delta \end{pmatrix} \circ f$~~

To compute in practice

Compute $\begin{pmatrix} 1+\alpha & \beta \\ \gamma & 1+\delta \end{pmatrix} \circ f - f$, and throw out second powers of Greek letters:

$$\begin{pmatrix} (1+\alpha)^3 - 1 & (1+\alpha)^2\beta & (1+\alpha)\beta^2 & \beta^3 \\ 3(1+\alpha)^2\gamma & (1+\alpha)^2(1+\delta) + 2(1+\alpha)\beta\gamma - 1 & 2(1+\alpha)\beta(1+\delta) + \beta\gamma^2 & 3\beta^2(1+\delta) \\ 3(1+\alpha)\gamma^2 & 2(1+\alpha)\gamma(1+\delta) + \beta\gamma^2 & (1+\alpha)(1+\delta)^2 + 2\beta\gamma(1+\delta) - 1 & 3\beta(1+\delta)^2 \\ \gamma^3 & \gamma^2(1+\delta) & \gamma(1+\delta)^2 & (1+\delta)^3 - 1 \end{pmatrix}$$

$\downarrow$ Throw away second powers

$$\begin{pmatrix} 3\alpha & \beta & 0 & 0 \\ 3\gamma & 2\alpha+\delta & 2\beta & 0 \\ 0 & 2\gamma & \alpha+2\delta & 3\beta \\ 0 & 0 & \gamma & 3\delta \end{pmatrix}.$$

You see why this is powerful. We've lost all the nonlinearity.

This is exactly the representation $gl(2) \to gl(4)$.

13.3

Now, compute $dp\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ 2x_2 \\ x_3 \\ 0 \end{pmatrix}$.

This is how a small change in g affects gv.

Similarly $dp\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ 2x_3 \\ 3x_4 \\ 0 \end{pmatrix}$

and with the other two we also get

$$\begin{pmatrix} 3x_1 & x_2 & 0 & 0 \\ 2x_2 & 2x_3 & 3x_2 & x_2 \\ x_3 & 3x_4 & 2x_3 & 2x_3 \\ 0 & 0 & x_4 & 3x_4 \end{pmatrix},$$

a matrix whose ~~disc~~ determinant is

$$3(x_2^2x_3^2 + 18x_1x_2x_3x_4 - 4x_1x_3^3 - 4x_2^3x_4 - 27x_1^2x_4^2)$$

$= 3\,\mathrm{Disc}(f)$.

~~In other words, the Lie algebra action is injective iff ... is nonsingular.~~

The point is, $G\vec{x}$ is 4-dimensional (depends on $\vec{x}$)

$\Longleftrightarrow$ a small neighborhood around $1 \in G$ is

~~...~~ $\Longleftrightarrow$ $g\vec{x}$ is.

So see what the four basis elts. of $\mathfrak{g}$ do to a generic vector.

12.5 (Review: 13.4)

Basic representation theory.

If G is a matrix Lie group, a (finite dim. cpx) representation of G is a Lie group homomorphism

$$\Pi : G \longrightarrow GL(V),$$

where V is a f.d. cpx vector space.

Same with reals.

Similarly, if $\mathfrak{g}$ is a (real & cpx) Lie algebra, a rep'n is a Lie alg hom

$$\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V).$$

If Π / π is 1-1, call it faithful.

Often write $g \cdot v$ instead of $\Pi(g)\, v$.

(Representations equivalent to linear cts group actions on vector spaces).

Example. (Do at board)

$GL(2)$ acting on binary cubic forms, via

$$g \circ f(u,v) = f((u,v)\, g).$$

12.6 = 13.5

Def. Let (Π, V) be a f.d. rep of a matrix Lie gp G.
A subspace W is invariant if $\Pi(A) w \in W$ for $w \in W$, $A \in G$.

(nontrivial if not W or 0)

A rep'n is irreducible if it has no nontrivial invariant subspaces.

Analogously for Lie algebra rep'ns.

Def. Given rep'ns (Π, V) and (Σ, W) of a matrix Lie group G.
A map $\phi : V \to W$ is an intertwining map of rep'ns if

(*) $$\phi(\Pi(A) v) = \Sigma(A) \phi(v)$$

for all $A \in G$, $v \in V$.

Sim: rep'n of a Lie algebra.

If ϕ is an vector space isomorphism also, we say it is an isomorphism of representations.

Problem. Classify all rep'ns of a Lie group up to iso!

Note, for (*), can also write

$$\phi(A \cdot v) = A \cdot \phi(v).$$

So ϕ commutes with "the" action of A.

Caution: There are two different actions!

12.7 = 13.6 = 14.1
(same as before!!)

Prop. Given $G, \mathfrak{g}$, a f.d. rep'n $\Pi : G \to GL(V)$.
Then $\exists$ a unique rep'n of $\mathfrak{g}$ acting on V with

$$\Pi(e^X) = e^{\pi(X)}$$

$$\pi(X) = \frac{d}{dt} \Pi(e^{tX}) \Big|_{t=0}$$

$$\pi(AXA^{-1}) = \Pi(A)\,\pi(X)\,\Pi(A)^{-1}$$

Proposition. (4.5)

(1) Let G be a conn'd Lie gp with Lie alg $\mathfrak{g}$.
Π a rep'n of G with π the assoc rep'n of $\mathfrak{g}$.
Then Π is irreducible $\iff$ π is.

(2) Again let G be a conn'd Lie group, Π_1, Π_2 rep'ns of it, π_1, π_2 associated Lie algebra rep'ns.
Then π_1 and π_2 are iso if and only if Π_1, Π_2 are.

This is nice.
(2) says: If π comes from a Lie group rep'n, then it's determined up to isomorphic.

Thm 5.6 (later). If G is simply connected, then π must come from a Lie group homomorphism (which is uniquely determined!)

13.7 = 14.2

Moral. If we want to understand the rep'n theory of simply connected Lie gps, just understand their Lie algebras.

Proof. (2) is an exercise! Prove (1).

Suppose first Π irreducible. Show π is.

Let $W \subseteq V$ be an invariant subspace.

Since G is conn'd, can write any $A \in G$ as

$$A = e^{X_1} e^{X_2} \cdots e^{X_m} \qquad (X_i \in \mathfrak{g})$$

Now W is invariant under all the $\pi(X_j)$.

Hence also each

$$\exp(\pi(X_j)) = I + \pi(X_j) + \frac{\pi(X_j)^2}{2} + \cdots$$

hence under

$$\Pi(A) = \Pi(e^{X_1} \cdots e^{X_m}) = \Pi(e^{X_1}) \cdots \Pi(e^{X_n})$$

$$= e^{\pi(X_1)} \cdots e^{\pi(X_n)}.$$

So W is invariant under $\Pi(G)$ in addition to $\pi(\mathfrak{g})$.

So it's $\{0\}$ or V.

Conversely, suppose π irreducible, W invariant for Π.
Then W is invariant under every vector of the form $\Pi(\exp tX)$ for $X \in \mathfrak{g}$,

hence under $\pi(X) = \frac{d}{dt} \Pi(e^{tX}) \Big|_{t=0}$.

Since π irred. W is $\{0\}$ or V.

Examples.

The standard representation.

A matrix Lie group $G \subseteq GL(n, \mathbb{C})$.

Just take the inclusion $G \hookrightarrow GL(n, \mathbb{C})$.

Similarly if $\mathfrak{g} \subseteq M_n(\mathbb{C})$. Just take the identity map.

The trivial representation.

$\Pi : G \to GL_1(\mathbb{C})$ $\qquad$ $\pi : \mathfrak{g} \to \mathfrak{gl}_1(\mathbb{C})$

$\Pi(A) = I \quad \forall A.$ $\qquad$ $\pi(A) = 0.$

Both irreducible

The adjoint representation.

$Ad : G \to GL(\mathfrak{g}) \qquad A \to Ad_A = \{ X \to AXA^{-1} \}$

$ad : \mathfrak{g} \to GL(\mathfrak{g}) \qquad A \to ad_A = \{ X \to [A, X] = AX - XA \}$

13.9 = 14.4

Let V_m = { homo polys of degree m in two (cpx) vars }

$$f(z_1, z_2) = a_0 z_1^m + a_1 z_1^{m-1} z_2 + \cdots + a_n z_2^m .$$

(dim $V_m = m+1$)

One rep'n of SL(2) or SU(2) we can get is

$$(\Pi_m(U) f)(z) = f(U^{-1} z) .$$

Alternatively,

$$(\Pi_m(U) f)(z) = f(U^T z) .$$

We have, with the first,

$$\Pi_m(U_1)\left(\Pi_m(U_2) f\right)(z) = \left(\Pi_m(U_2) f\right)(U_1^{-1} z)$$

$$= (U_2^{-1} U_1^{-1} z)$$

$$= ((U_1 U_2)^{-1} z)$$

$$= (\Pi_m(U_1 U_2) f)(z) .$$

Same if you put transposes.

Later: (1) These are irreducible;

(2) every f.d. rep'n of SU(2) is iso to one of these.

How to compute the associated rep'n π_m of $s\ell(2)$?

$$(\pi_m(X) f)(z) = f(e^{-tX} z)\Big|_{t=0}.$$

Make sure you understand why the parens are where they are!!

Write $z(t) = (z_1(t), z_2(t))$ be the curve in $\mathbb{C}^2$

$$z(t) = e^{-tX} z.$$

We have

$$\pi_m(X) f = \left(\frac{\partial f}{\partial z_1} \frac{dz_1}{dt} + \frac{\partial f}{\partial z_2} \frac{dz_2}{dt} \right)\Big|_{t=0}$$

Now, $\left.\frac{dz}{dt}\right|_{t=0} = -Xz$

So above is

$$-\frac{\partial f}{\partial z_1}(X_{11} z_1 + X_{12} z_2) - \frac{\partial f}{\partial z_2}(X_{21} z_1 + X_{22} z_2)$$

Choose the following basis for $s\ell(2)$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\pi_m(H) = -z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}.$$

$$\pi_m(X) = -z_2 \frac{\partial}{\partial z_1}$$

$$\pi_m(Y) = -z_1 \frac{\partial}{\partial z_2}$$

These are endomorphisms of V_m.

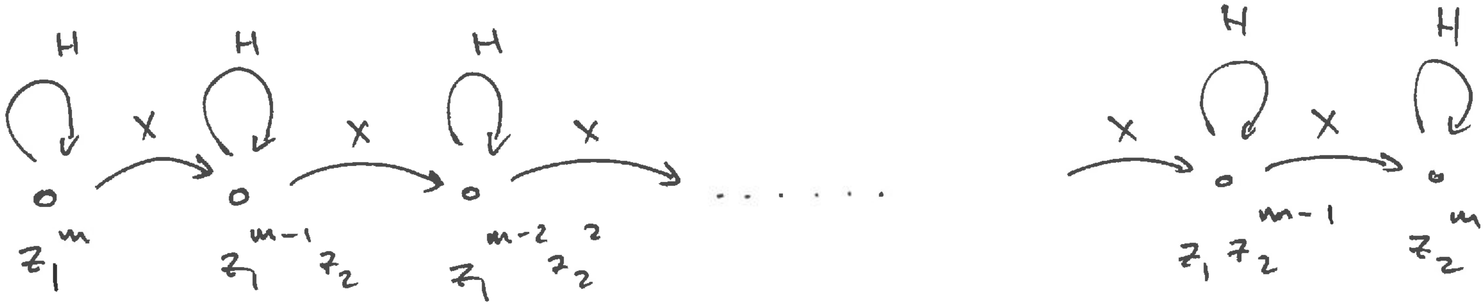
14.6

What do these do to a basis elt. $z_1^{m-k} z_2^{k}$ for V_m?

$$\pi_m(H)(z_1^{m-k} z_2^{k}) = (-m+2k)\, z_1^{m-k} z_2^{k}$$

$$\pi_m(X)(z_1^{m-k} z_2^{k}) = -(m-k)\, z_1^{m-k-1} z_2^{k+1}$$

$$\pi_m(Y)(z_1^{m-k} z_2^{k}) = -k\, z_1^{m-k+1} z_2^{k-1}.$$



__All__ of them are eigenvectors for $\pi_m(H)$ w/ eigenvalue $-m+2k$.

(zero __only__ if $m = 2k$, so $z_1^{m/2} z_2^{m/2}$ w/ m even.)

$\pi_m(X)$ shifts you to the right one. (adds 1 to k)
If you're already at the right, $m = k$ and $\pi_m(X)$ acts by 0.

$\pi_m(Y)$ shifts you to the left one.
If you're already at the left, $k = 0$ and $\pi_m(X)$ acts by 0.

__Also__: $\pi_m(X)$ __increases__ eigenvalue of $\pi_m(H)$ by 2
$\pi_m(Y)$ __decreases__ eigenvalue of $\pi_m(H)$ by 2.

14.7

Proposition. The representation Π of $SL(2)$ is irreducible.

Proof. It suffices to show the representation π of $sl(2)$ is irreducible.

But now you can play games with the diagram. (See Hall, p. 84 for a more formal proof.)

Suppose W is an invariant subspace and $0 \neq w \in W$. Hit it with $\pi_m(X)$ a bunch of times until you get a nonzero multiple of z_2^m. That's in W.

Now keep marching to the left.

(15.1 =)

In the book:

Direct sums. If G acts on V_1, V_2 it acts on $V_1 \oplus V_2$ by each component.

Same with $\mathfrak{g}$.

Tensor products. pp 85 – 7 Review of construction.

Two constructions.

(1) Given a rep'n (Π_1, U) of G
(Π_2, V) of H

Get a rep'n $(\Pi_1 \otimes \Pi_2, U \otimes V)$ of $G \times H$

$$(\Pi_1 \otimes \Pi_2)(A, B) = \Pi_1(A) \otimes \Pi_2(B)$$

for $A \in G$ and $B \in H$.

(2) If both are rep'ns of G

$$(\Pi_1 \otimes \Pi_2)(A) = \Pi_1(A) \otimes \Pi_2(A).$$

15.2

Proposition. Given rep'ns Π_1 of G, Π_2 of H

$\Pi_1 \otimes \Pi_2$ of $G \times H$.

If $\pi_1 \otimes \pi_2$ is the associated rep'n of $\mathfrak{g} \oplus \mathfrak{h}$, then

$$(\pi_1 \otimes \pi_2)(X, Y) = \pi_1(X) \otimes I + I \otimes \pi_2(X),$$

for all $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$.

Proof. If $u(t)$ smooth curve in U, $v(t)$ in V, the product rule reads

$$\frac{d}{dt}\left(u(t) \otimes v(t)\right) = \frac{du}{dt} \otimes v(t) + u(t) \otimes \frac{dv}{dt}$$

and so

$$(\pi_1 \otimes \pi_2)(X, Y)(u \otimes v)$$

$$= \frac{d}{dt} \Pi_1(e^{tX})\, u \otimes \Pi_2(e^{tY})\, v \Big|_{t=0}$$

$$= \left(\frac{d}{dt} \Pi_1(e^{tX})\, u \Big|_{t=0}\right) \otimes v + u \otimes \left(\frac{d}{dt} \Pi_2(e^{tY})\, v \Big|_{t=0}\right).$$

In general, given rep's (π_1, U) of $\mathfrak{g}$ and (π_2, V) of $\mathfrak{h}$,

get a rep'n $\pi_1 \otimes \pi_2$ of $\mathfrak{g} \oplus \mathfrak{h}$ on $U \otimes V$:

$$(\pi_1 \otimes \pi_2)(X, Y) = \pi_1(X) \otimes I + I \otimes \pi_2(Y).$$

This is linear in (X, Y).

$\pi_1(X) \otimes \pi_2(Y)$ is not!

15.3

Dual representations.

Given V, let $V^* = \mathrm{Hom}(V, k)$ be its dual space. (← whatever field V is defined over)
(linear fns. from V to k)

Given $A \in GL(V)$, define $A^T \in GL(V^*)$ by

$$(A^T \phi)(v) = \phi(Av) \quad \text{for each } \phi \in V^*, v \in V.$$

If $v_1, \ldots, v_n$ is a basis, define the dual basis $\phi_1 \ldots \phi_k$ by $\phi_j(v_k) = \delta_{jk}$

so that you get an isomorphism $V \longrightarrow V^*$
$v_i \longrightarrow \phi_i$.

(Warning. This assumes V is finite dimensional. Weird shit happens otherwise.)

Exercise. The matrix A^T is the usual matrix transpose of A.

Claim. $(AB)^T = B^T A^T$.

Crap Proof. Write it out using matrix coefficients. (Please DON'T actually do this)

Better proof. For all ϕ and v,

$$((AB)^T \phi)(v) = \phi(ABv)$$

$$(B^T(A^T\phi))(v) = (A^T\phi)(Bv) = \phi(ABv).$$

Def. Given a rep'n (π, V) of G, the dual (contragredient) rep'n is the rep. (π^*, V^*) on the dual space given by

$$\pi^*(g) = (\pi(g^{-1}))^T.$$

Note that inverses and transposes both flip order, so we're straightened out here.

15.4

Check, we get the assoc. rep'n $\pi^*(X) = -\pi(X)^T$.

<u>Proposition / Exercise.</u> Given (Π, G), Π^* irred $\Leftrightarrow$ Π irred,

$$(\Pi^*)^* \cong \Pi.$$

Similarly for Lie alg. rep'ns.

<u>Complete reducibility.</u>

<u>Def.</u> A f.d. rep'n of a group or Lie algebra is <u>completely reducible</u> if it is iso to a direct sum of finitely many irreducible rep'ns.

(So, in particular, irreducible rep'ns are completely reducible. Yeah, I know.)

We can understand the irred. rep'ns, so if all rep'ns of a given Lie group are CR then this is really nice. This happens if the Lie group is "reductive".

<u>Ex.</u> Given $\Pi : \mathbb{R} \to GL(2, \mathbb{C})$

$$x \to \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

This is <u>not</u> completely reducible.

Can check, The invariant subspaces are $\{0\}$, $\mathrm{Span}(v_1)$, and V.

15.5 = 16.1

Proposition. Let V be a c.r. rep'n of a group or Lie algebra. Then

(1) For every invariant $U \subseteq V$ there is an invariant complement W with $V = U \oplus W$.

(2) Every invariant subspace of V is CR.

Proof. (1) Write $V = U_1 \oplus \cdots \oplus U_k$ irred. invariant.

Given $U \neq V$, there is some $U_{j_1} \not\subseteq U$, and so with $U_{j_1} \cap U = \{0\}$.

Either $U_{j_1} + U = V$, sum is direct, and we're done, or $U_{j_1} + U$ doesn't contain some U_{j_2}. Etc. Keep going.

(2) If $U \subseteq V$ invariant, first establish ~~existence of complement~~ property of (1) for U.

Suppose $X \subseteq U$ invariant.

By (1), can write $V = X \oplus Y$, Y also invariant

Write $Z := Y \cap U$ invariant and show $U = X \oplus Z$.

Write any $u \in U$ as $u = x + y$ $\quad (x \in X, y \in Y)$

but $y = u - x \in U - U = U$

so $y \in Z$ done.

So, finally, if U is not already irreducible, find some invariant subspace X with $U = X \oplus Z$

X, Z invariant

If these are not irreducible, decompose further.

Eventually we win.

15.6

Prop. If G mat Lie gp, (Π, V) a f.d. unitary rep'n of G,
then Π is completely reducible.

Similarly, if $\mathfrak{g}$ is a real Lie algebra and π is f.d. "unitary" $(\pi(X)^* = -\pi(X))$
then π is completely reducible.

What does unitary mean? That $\Pi(g) \in U(V)$ for all g.

Proof. Let $W \in W$ be invariant and write $V = W \oplus W^\perp$
orthogonal complement wrt the inner product

Now by unitoriness, $\Pi(A)^* = \Pi(A)^{-1} = \Pi(A^{-1})$ for $A \in G$

So for all $w \in W$, $v \in W^\perp$,

$$\langle \Pi(A) v, w \rangle = \langle v, \Pi(A)^* w \rangle = \langle v, \Pi(A^{-1}) w \rangle$$
$$= \langle v, \text{smth in } W \rangle$$
$$= 0.$$

So $W^\perp$ must also be invariant.

Now apply the same trick.
$$V = W \oplus W^\perp$$
If either is not irreducible, find an invariant subspace
and break up in the same way.
Eventually this stops.

(Argument for π's similar.)

Theorem. If G is a compact matrix Lie group, every f.d.
rep'n is completely reducible

Proof. Cook up a weird inner product with respect to
which it's unitary. See pp. 92-94.

15.7

Schur's lemma.

(1) Let V, W irred (real or cpx) rep'ns of a group or Lie algebra, and $\phi : V \to W$ an intertwining map.

Then ϕ is 0 or an iso.

(2) Given V complex, and $\phi : V \to V$, then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$.

(3) Given V, W cpx as above with ϕ_1, ϕ_2 nonzero intertwining maps $V \to W$. Then $\phi_1 = \lambda \phi_2$ for some $\lambda \in \mathbb{C}$.

Cor. #1. Let Π be a complex irrep (irreducible representation) of a mat. Lie gp G. If $A \in Z(G)$ (the center) then $\Pi(A) = \lambda I$ for some $\lambda \in \mathbb{C}$.

Similarly if π is a cpx irrep of $\mathfrak{g}$ with $A \in Z(\mathfrak{g})$, then $\pi(A) = \lambda I$.

Proof. (for group case; Lie algebra case similar)

Since $\Pi(A)\Pi(B) = \Pi(B)\Pi(A)$, $\Pi(A)$ is an intertwining map of the space with itself!

Cor. #2. A complex irrep of a commutative group or Lie algebra must be one-dimensional.

Proof. Since $\Pi(A) = \lambda A$ for each A, every subspace of V is invariant.

15.8

Proof of Schur (Group case).

(1) Let $v \in \ker(\phi)$. Then

$$\phi(\Pi(A)v) = \Sigma(A)\phi(v) = 0.$$

So $\ker \phi$ is invariant. DONE

Well, almost: it's zero or one-to-one

and in the latter case $\mathrm{Im}(\phi)$ is invariant.

For all $w = \phi(v)$,

$$\Sigma(A)w = \Sigma(A)\phi(v) = \phi(\Pi(A)v).$$

(2) Given $\phi: V \to V$ with $\phi\Pi(A) = \Pi(A)\phi$.

Now ϕ has an eigenvalue $\lambda \in \mathbb{C}$ w/ eigenspace U.

So U is an invariant subspace, hence $U = V$.

(3) $\phi_1 \circ \phi_2^{-1}$ intertwining map $V \to V$. Use (2).

All rep'ns of $sl(2, \mathbb{C})$:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

So the matrix of $\mathrm{ad}(H)$ is $\begin{bmatrix} 0 & & \\ & 2 & \\ & & -2 \end{bmatrix}$.

We already had the representations of binary n-ic forms discussed before.

Departing from Hall, write

$$\underline{\mathrm{Sym}^n(\mathbb{C}^2)} = \{\text{binary cubic forms of deg } n\}.$$

Then, for each $n \geq 1$, $\mathrm{Sym}^n(\mathbb{C}^2)$ is an irrep of dim $n+1$.

Theorem. Every irrep of $\mathfrak{sl}(2,\mathbb{C})$ is one of these.

Proof. Given an irrep (π, V) . . .

Lemma. Let u be an eigenvector of $\pi(H)$ with EV $q \in \mathbb{C}$.

Then

$$\pi(H)\pi(X)\, u = (q+2)\,\pi(X)\, u.$$

Proof.

$$\begin{aligned}\pi(H)\pi(X)\, u &= \pi(X)\pi(H)\, u + [\pi(H), \pi(X)]\, u \\ &= \pi(X)\pi(H)\, u + \pi([H, X])\, u \\ &= \pi(X) \cdot q u + 2\pi(X)\, u.\end{aligned}$$

So: $\pi(X)$ sends eigenvectors to eigenvectors, and raises the EV by 2.

Similarly, $\pi(H)\pi(Y)\, u = (q-2)\,\pi(Y)\, u$.

Proof of theorem. Given an irrep (π, V) of $\mathfrak{sl}(2,\mathbb{C})$.

Let u be an eigenvector for $\pi(H)$ (it must have one!) with eigenvalue q.

Then by lemma, $\pi(H)\pi(X)^k u = (q+2k)\,\pi(X)^k u$.

We can't have infinitely many eigenvalues!

So, for some $N \geq 0$,

$$u_0 := \pi(X)^N u \neq 0, \quad \pi(X)^{N+1} u = 0.$$

Write $\lambda = q + 2N$,

$$\pi(H)\, u_0 = \lambda u_0 \qquad \pi(X)\, u_0 = 0.$$

15.10 = 16.6

Write now, for each k $u_k = \pi(Y)^k u_0$, with $\pi(H) u_k = (\lambda - 2k) u_k$

Can check: $\pi(X) u_k = k(\lambda - (k-1)) u_{k-1}$ for all $k \geq 1$.

Let u_m be the last nonzero one. (Same argument as before)

Now $0 = u_{m+1} = \pi(X) u_{m+1} = (m+1)(\lambda - m) u_m$ so $\lambda = m$.

We have thus listed basis vectors for an invariant subspace of V
(by irreducibility, is V itself)

Have to be linearly independent since they are EV's of $\pi(H)$ with distinct eigenvalues.

But we've just written down the entire representation.

Conversely, check that we really do have a representation.
Use our earlier construction, or define $\pi(H)$, $\pi(X)$, $\pi(Y)$ by the relations above and check the commutators.

16.7 A little plethysm. (See Fulton-Harris, Rep Thy, Ch 11)

Apparently this is a word. Analyze decompositions of rep'ns.

Example. Let $V \cong \mathbb{C}^2$ be the standard repn. of $sl(2)$.

i.e. $\pi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\begin{pmatrix} r \\ s \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$.

We have been discussing the symmetric power rep'ns $Sym^n(V)$.

e.g. $Sym^2(V) = \mathrm{Span}\{x^2, xy, y^2\}$.

$V \otimes V = \mathrm{Span}\{x \otimes x, x \otimes y, y \otimes x, y^2\}$

and $Sym^2(V) := V \otimes V / \langle v \otimes w - w \otimes v \rangle$.

We get a rep'n of $\circlearrowleft$ on $V \otimes V$:

~~$\pi(g)(v \otimes w) = (\pi(g)v) \otimes (\pi(g)w)$.~~ GAAAH!!! TOTALLY WRONG!

correct: $\pi(g)(v \otimes w) = \pi(g)v \otimes w + v \otimes \pi(g) \cdot w$.

This factors through the quotient above.

Since $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$, $x \to ax + cy$

$y \to bx + dy$

and so $\{x^2, xy, y^2\}$

$\downarrow$

~~$\{(ax+cy)^2, (bx+dy)(ax+cy), (bx+dy)^2\}$.~~

16.8

$x^2 \to (ax+cy)\cdot x + x\cdot(ax+cy)$
$= 2x(ax+cy)$

$xy \to (ax+cy)\cdot y + x\cdot(bx+dy)$

This is all a bit weird.

So ask. what does $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ do?

Or, more precisely, $\pi(H)$

$H(x\cdot x) = x\cdot H(x) + H(x)\cdot x = 2x^2$.
$H(x\cdot y) = x\cdot H(y) + H(x)\cdot y = xy - xy = 0$
$H(y\cdot y) = y\cdot H(y) + H(y)\cdot y = -2y^2$.

Similarly, $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ sends $y \to x$ and kills x.

$X(x\cdot x) = x\cdot X(x) + X(x)\cdot x = 0$
$X(x\cdot y) = x\cdot X(y) + X(x)\cdot y = x^2$
$X(y^2) = y\cdot X(y) + X(y)\cdot y = 2xy$.

Similarly with $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

So the point is these natural basis are all eigenvectors

	$\{x^2,$	$xy,$	$y^2\}$
EV	2	0	-2

(x-exponent) − (y-exponent)

$\mathrm{Sym}^3 V = \mathrm{Span}\{x^3, x^2y, xy^2, y^3\}$

3 1 -1 -3

16.9

$\mathrm{Sym}^4 V = \mathrm{Span}\{x^4, x^3 y, x^2 y^2, x y^3, y^4\}$

4 2 0 -2 4

What about $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$?

If the 3 basis vectors of $\mathrm{Sym}^2 V$ are v_1, v_2, v_3 then this is spanned by $\left\{ \begin{matrix} v_1 v_2, & v_1 v_3, & \text{and} & v_2 v_3 \\ v_1^2, & v_2^2 & & v^3 \end{matrix} \right\}$.

So:

$$\mathrm{Sym}^2(\mathrm{Sym}^2 V) = \mathrm{Span}\{x^2 \cdot x^2,\ x^2 \cdot xy,\ x^2 \cdot y^2,\ xy \cdot xy,\ xy \cdot y^2,\ y^3\}$$

It is 6-dimensional, and we get a natural surjection

$$\mathrm{Sym}^2(\mathrm{Sym}^2 V) \longrightarrow \mathrm{Sym}^4 V$$

$$q_1 \cdot q_2 \longrightarrow q_1 q_2$$

whose kernel is spanned by $x^2 \cdot y^2 - xy \cdot xy$.

(Does your head hurt yet?)

Can we figure out the eigenvalues directly?

If v_1 and $v_2 \in \mathrm{Sym}^2 V$ are EV's of H with EV λ_1, λ_2,

$$H(v_1 \cdot v_2) = v_1 \cdot H(v_2) + H(v_1) \cdot v_2$$

$$= \lambda_2 v_1 \cdot v_2 + \lambda_1 v_1 \cdot v_2$$

$$= (\lambda_1 + \lambda_2) v_1 \cdot v_2.$$

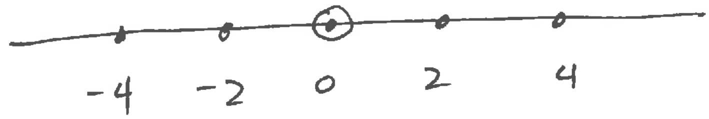
So: $-2 + -2$, $-2 + 0$, $-2 + 2$,

$0 + 0$, $0 + 2$, $2 + 2$.

16.10

• : multiplicity 1
⊙ : multiplicity

Eigenvalues of $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$



Now by general theory (unproved so far)

$\mathfrak{sl}(2)$ is <u>simple</u> (no ideals)

hence semi<u>simple</u> (direct sum of simples)

hence reductive.

So $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is a direct sum of irreducibles,

and we know <u>all irreducibles are $\mathrm{Sym}^k V$</u>

<u>and can be read off from their eigenvalues.</u>

Here just $\mathrm{Sym}^2(\mathrm{Sym}^2 V) = \mathrm{Sym}^4 V \oplus \underbrace{\mathrm{Sym}^0 V}_{\text{Trivial rep'n}}$.

Ex. <u>Verify this!</u>