5.6 = 6.1

Lie groups in general.

Def. A topological space X is a (real) manifold of dimension n if for each $x \in X$ $\exists$ $x \in U \subseteq X$ and a homeomorphism $U \xrightarrow{\phi_x} \mathbb{R}^n$.

It is smooth if there is an atlas of U covering X such that

$$\mathbb{R}^n \xrightarrow{\phi_U^{-1}} \underbrace{U \longrightarrow U \cap U'}_{\substack{\text{abuse of notation.}\\ \text{Only get a}\\ \text{map from part of } \mathbb{R}^n}} \xrightarrow{\phi_{U'}} \mathbb{R}^n$$

is smooth.

Also demand that X is Hausdorff and "second countable".
(countable basis)

Def. A Lie group is a smooth manifold G which is also a group s.t. the maps

$$G \times G \longrightarrow G$$
$$(g_1, g_2) \longmapsto g_1 g_2$$

$$G \longrightarrow G$$
$$g \longmapsto g^{-1}$$

are smooth.

Example. $GL_n(\mathbb{R})$ clearly is. (of dim n^2.)
Any matrix Lie group is.
e.g. why is $SL_n(\mathbb{R})$? (implicit function theorem)
(will come back later).

6.2

Ch 2.1 The exponential map.

Def. If $X \in M_{n\times n}(\mathbb{C})$, its exponential e^X or $\exp(X)$ is

$$\exp(X) := \sum_{m=0}^{\infty} \frac{X^m}{m!}$$

Proposition. The series converges for all $X \in M_n(\mathbb{C})$ and exp is a continuous function.

Def. For $X \in M_n(\mathbb{C})$, its Hilbert - Schmidt norm $\|X\|$ is defined by

$$\|X\|^2 = \sum_{j,k=1}^{n} |x_{j,k}|$$

exercise!

$$= \operatorname{tr}(X^* X)$$

This comes from the Hilbert - Schmidt inner product

$$\langle A, B \rangle = \operatorname{tr}(A^* B).$$

Exercises. This norm satisfies:

(1) $\|X + Y\| \le \|X\| + \|Y\|$

(2) $\|XY\| \le \|X\| \, \|Y\|$

(3) For a seq of matrices $\{X_m\}$,

$X_m \to X$ entrywise

iff

$\|X_m - X\| \to 0$.

6.3

(Ch 2.1 p.2)

Proof of convergence and continuity of exp.

Convergence.

$$\sum_{m=0}^{\infty} \left\| \frac{X^m}{m!} \right\| = \sum_{m=0}^{\infty} \frac{\|X^m\|}{m!} \le \sum_{m=0}^{\infty} \frac{\|X\|^m}{m!} = \exp(\|X\|).$$

Continuity.

First of all, the function $X \to X^n$ is cts for each X.

Now, apply the Weierstrass M-test or similar.

Proposition.

(1) $e^0 = I$.

(2) $(e^X)^* = e^{X^*}$.

(3) e^X is invertible, and $(e^X)^{-1} = e^{-X}$.

(4) $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X}$ for any $\alpha, \beta \in \mathbb{C}$.

(5) $e^{X+Y} = e^X e^Y = e^Y e^X$ if X and Y commute

(6) If C is invertible, $e^{CXC^{-1}} = Ce^XC^{-1}$.

Proof. (1) obvious, (2) obvious when you write it out.

(3), (4) follow from (5)

For (5), write out both sides, we have

$$(X+Y)^m = \sum_{k=0}^{m} \binom{m}{k} X^k Y^{m-k} \quad \text{if } X, Y \text{ commute.}$$

(6) Use $(CXC^{-1})^m = CX^mC^{-1}$

Prop. Let X be a $n \times n$ complex matrix.

Then the function
$$\mathbb{R} \to M_n(\mathbb{C})$$
$$t \to e^{tX}$$
is smooth (a smooth curve, by def'n)

and
$$\frac{d}{dt} e^{tX} = X e^{tX} = e^{tX} X$$

In particular,
$$\frac{d}{dt}(e^{tX})\Big|_{t=0} = X.$$

Note: $\frac{d}{dt}(e^{X+tY})$ is not necessarily $e^{X+tY} Y$.

Noncommutativity makes things fun.

Proof. Differentiate the power series term by term.

(Exercise: check the details.)

Computing the matrix exponential.

Idea. Use $e^{CXC^{-1}} = C e^X C^{-1}$.

If $X = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ then $e^X = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}$.

Otherwise.

Theorem. Every matrix A can be written uniquely as

$A = S + N$ where: $\begin{cases} S \text{ is diagonalizable} \\ N \text{ is nilpotent} \\ SN = NS. \end{cases}$

6.5

Proof, without uniqueness.

Jordan form it! $A = CXC^{-1}$ $X = \begin{bmatrix} \lambda_1 & 1 & & & & \\ & \ddots & 1 & & & \\ & & \lambda_1 & & & \\ & & & \lambda_2 & 1 & \\ & & & & \ddots & 1 \\ & & & & & \lambda_2 \\ & & & & & & \text{etc.} \end{bmatrix}$

Write $X = \underbrace{S}_{\text{lambda part}} + \underbrace{N}_{\text{one part}}$

Then N is nilpotent because $Ne_j = e_{j-1}$ or 0.

S and N commute because: check on blocks

S diagonal on blocks!

And $A = CSC^{-1} + \underbrace{CNC^{-1}}_{\text{nilpotent if } N \text{ is!}}$

$(CNC^{-1})^k = 0 \Longleftrightarrow N^k = 0.$

Why does this help?

Example. Let $X_1 = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$.

Check. Eigenvectors are $(1, i)$ w/ eigenvalue $-ia$

$(i, 1)$ " ia

So $X_1 = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} -ia & 0 \\ 0 & ia \end{pmatrix}\begin{pmatrix} 1/2 & -i/2 \\ -i/2 & 1/2 \end{pmatrix}$

$e^{X_1} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix}\begin{pmatrix} 1/2 & -i/2 \\ -i/2 & 1/2 \end{pmatrix}$

$= \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}.$

~~Ex~~ Here $\sum_{n=0}^{\infty} \frac{1}{n!} CX^n C^{-1} = C\left(\sum_{n=0}^{\infty} \frac{1}{n!} X^n\right) C^{-1}.$

6.6.

Let $X_2 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. Then $e^{X_2} = I + X_2 + \frac{X_2^2}{2}$.

$$X_3 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

Then $\exp(X_3) = \exp\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) \exp\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right)$.

Reduce to above two cases.

Application to first-order ODEs:

Given $\frac{d\vec{v}}{dt} = X\vec{v}$ $\quad \vec{v}(t) \in \mathbb{R}^n$

$\vec{v}(0) = \vec{v}_0$

The unique solution is $\vec{v}(t) = e^{tX}\vec{v}_0$.

Matrix logarithms.

Proposition. The function $\mathbb{C} \to \mathbb{C}$ $\log z := \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(z-1)^m}{m}$

(1) is defined and analytic in a circle of radius 1 about $z = 1$

(2) satisfies $e^{\log z} = z$ when $|z - 1| < 1$

(3) satisfies $\log e^u = u$ when $|u| < \log 2$ (so that $|e^u - 1| < 1$).

Proof. Taking analysis qual? → Yes → Read p. 37 very carefully

→ No → Smile and nod.

6.7.

Def. For $A \in M_n(\mathbb{C})$,

$$\log A := \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-I)^m}{m}$$

whenever the series converges.

By what we have seen, definitely converges if $\|A - I\| < 1$.
But might converge otherwise too (e.g. if $A - I$ nilpotent series will be finite)

Theorem. (1) It's defined and continuous when $\|A - I\| < 1$

(2) when $\|A - I\| < 1$, $e^{\log A} = A$

(3) when $\|X\| < \log 2$, $\|e^X - I\| < 1$
$\log(e^X) = X$.

Proof Discussed (1) already.

(2). If $\|A - I\| < 1$,

Case 1. A diagonalizable. $A = CDC^{-1}$,

$$(A - I)^m = C \begin{pmatrix} (z_1 - 1)^m & & \\ & \ddots & \\ & & (z_n - 1)^m \end{pmatrix} C^{-1}.$$

Exercise. Since $\|A - I\| < 1$, all eigenvalues satisfy $|z_j - 1| < 1$.

$$\text{So} \quad \log D = \begin{pmatrix} \log z_1 & & \\ & \ddots & \\ & & \log z_n \end{pmatrix}$$

and $\log(CDC^{-1}) = C\log(D)\,C^{-1}$

So $e^{\log A} = C \begin{pmatrix} e^{\log \lambda_1} & & \\ & \ddots & \\ & & e^{\log \lambda_n} \end{pmatrix} C^{-1} = A$.

If A is not diagonalizable, use a topological argument.
There exists a seq $A_m \to A$ of diagonalizable matrices
use continuity of logarithm and exponential maps.

How can we see?
A diagonalizable $\longleftarrow$ eigenvalues are distinct
$\Updownarrow$
charpoly(A) has distinct roots
$\Updownarrow$
$\mathrm{Disc}(\text{charpoly}(A)) \neq 0$.

If a polynomial is 0 on an open nbd. of $\mathbb{C}^n$, then it is identically 0.

(3) is done seriously.

Proposition. There is a constant c s.t. for all $B \in M_n(\mathbb{C})$ with $\|B\| < \frac{1}{2}$,

$$\|\log(I+B) - B\| \le c\|B\|^2.$$

Equivalently, $\log(I+B) = B + O(\|B\|^2)$.

6.9

Proof. $\log(I+B) - B = \sum_{m=2}^{\infty} (-1)^{m+1} \frac{B^m}{m}$

$$= B^2 \sum_{m=2}^{\infty} (-1)^{m+1} \frac{B^{m-2}}{m}$$

$$\|\cdot\| \le \|B\|^2 \cdot \underbrace{\sum_{m=2}^{\infty} (-1)^{m+1} \frac{\|B\|^{m-2}}{m}}$$

$$\le \|B\|^2 \cdot \underbrace{\sum_{m=2}^{\infty} (-1)^{m+1} \frac{\left(\frac{1}{2}\right)^{m-2}}{m}}_{\text{Call this } c.}$$

Thm. Every $X \in GL_n(\mathbb{C})$ is e^A for some matrix A.

7.1. [First do: Prop. in 6.4, then 6.8 - 6.9]

Theorem. (Lie Product Formula)

For $X, Y \in M_n(\mathbb{C})$, $e^{X+Y} = \lim_{m\to\infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m$.

Proof. By expanding the power series,

$$e^{\frac{X}{m}} e^{\frac{Y}{m}} = I + \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)$$

and for large m, $e^{\frac{X}{m}} e^{\frac{Y}{m}}$ is in the domain of log.

So $\log\left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right) = \log(\text{above})$

$$= \frac{X}{m} + \frac{Y}{m} + O\left(\left\| \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)\right\|^2\right)$$

$$= \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)$$

and so $e^{\frac{X}{m}} e^{\frac{Y}{m}} = \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)$

$$\left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m = \exp\left(X + Y + O\left(\frac{1}{m}\right)\right).$$

So: $\lim_{m\to\infty}\left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m = \lim_{m\to\infty} \exp\left(X + Y + O\left(\frac{1}{m}\right)\right)$

$$= \exp\left(X + Y + \lim_{m\to\infty} O\left(\frac{1}{m}\right)\right) \quad \text{exp(-) is continuous}$$

$$= \exp(X+Y).$$

Later (Baker-Campbell-Hausdorff formula). Don't take a limit; see what you get.

7.2

Theorem. For $X \in M_n(\mathbb{C})$,

$$\det(e^X) = e^{\operatorname{trace}(X)}.$$

Proof. Suppose first X is diagonalizable.

Since $\det(e^{CDC^{-1}}) = \det(C e^D C^{-1})$
$= \det(e^D)$

$$e^{\operatorname{trace}(CDC^{-1})} = e^{\operatorname{tr}(C)},$$

can just take $X = D$ diagonal.

$$\det \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} = \exp\left(\operatorname{tr} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right)$$

If X is not diagonalizable, use a continuity argument.

One Parameter Subgroups.

Def. A function $A : \mathbb{R} \to GL(n, \mathbb{C})$ is a one parameter subgroup if

(1) it is continuous

(2) $A(0) = I$

(3) $A(t+s) = A(t)\, A(s)$ for all $t, s \in \mathbb{R}$.

Example. (1) $\mathbb{R} \to SO(2)$ $\quad t \to \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$.

(2) Fix any $X \in M_n(\mathbb{C})$, then $t \to e^{tX}$.

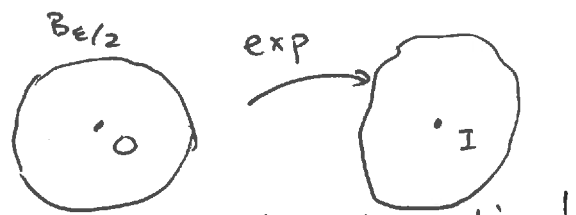
7.3. **Theorem.** If $A(\cdot) \subseteq GL_n(\mathbb{C})$ is a 1PS, then it is of the form

$$A(t) = e^{tX} \quad \text{for some } X \in M_n(\mathbb{C}).$$

(Note: X is unique, since $X = \frac{d}{dt} A(t)\big|_{t=0}$.)

Lemma. Let $\begin{cases} \varepsilon < \log 2 \\ B_{\varepsilon/2} \text{ ball of radius } \varepsilon/2 \text{ around } 0 \in M_n(\mathbb{C}) \\ U = \exp(B_{\varepsilon/2}) \end{cases}$

Then every $B \in U$ has a unique square root $C \in U$, given by $C = \exp(\frac{1}{2} \log B)$.



cts. bijection!

Proof. Clear: $C^2 = B$, $C \in U$. Less clear: uniqueness.

If also $(C')^2 = B$, write $Y = \log C'$, $\exp(Y) = C'$

$$\exp(2Y) = (C')^2 = \exp(\log B).$$

Now $Y \in B_{\varepsilon/2}$ and $2Y \in B_\varepsilon$

also $\log B \in B_{\varepsilon/2} \subseteq B_\varepsilon$.

By previous results, exp is injective on B_ε

$$\exp(2Y) = \exp(\log B)$$
$$\Rightarrow Y = \frac{1}{2} \log B.$$

so $C' = \exp(\frac{1}{2} \log B) = C$.

7.4

Proof of theorem.

Let U be as above.

Choose $t_0 > 0$ with $A(t) \in U$ for $|t| \le t_0$.

Write $X = \frac{1}{t_0} \log A(t_0)$.

Then $t_0 X = \log A(t_0) \in B_{\varepsilon/2}$, $e^{t_0 X} = A(t_0)$.

Now: $A(t_0)$ has a unique square root in U, and $e^{t_0 X/2}$ is a square root.

Also, $A(\frac{t_0}{2})$ is by def. of 1PS.

So $A(\frac{t_0}{2}) = e^{t_0 X/2}$.

Conclude: $A(\frac{t_0}{2^k}) = e^{t_0 X/2^k}$ by same reasoning

$$A\left(\frac{m t_0}{2^k}\right) = \exp\left(\frac{m t_0}{2^k} X\right)$$

So: $A(t) = \exp(tX)$ for all t of the form $t = \frac{m t_0}{2^k}$.

Since $\exp(tX)$ and $A(t)$ are continuous and agree on a dense subset of $t \in \mathbb{R}$, they agree.

[§2.5. Polar decomposition. Cool; might go back.]

7.5 = 8.1 (review)

Lie algebras.

Def. A (real / complex) Lie algebra is a (real / complex) vector space $\mathfrak{g}$ (Gothic g, I use underlines) with a non-associative product $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

(1) bilinearity;

(2) skew-symmetry: $[X, Y] = -[Y, X]$ for $X, Y \in \mathfrak{g}$.

(3) the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Notation / Remarks. (1) $[\cdot,\cdot]$ is the Lie bracket

(2) $[X, X] = 0$ always

(3) X and Y commute if $[X, Y] = 0$

(4) the Lie algebra is abelian if $[X, Y] = 0 \ \forall X, Y$.

Examples of Lie algebras.

- Trivial.
- Any of dimension 2?
- The cross product Lie algebra.

  (Prove: By bilinearity, check for $\vec{i}, \vec{j}, \vec{k}$. Do it!)

- Let A be any associative algebra (e.g. $M_n(\mathbb{C})$ or a subalgebra)

  Then $\mathfrak{g} = A$ with $[X, Y] = XY - YX$.

  (or: $\mathfrak{g}$ any subalgebra with ~~Ex~~ $XY - YX \in \mathfrak{g}$ for all $x, y \in \mathfrak{g}$.)

7.6 = 8.2

Ex. Let $\mathfrak{sl}(n, \mathbb{C}) := \{ X \in M_n(\mathbb{C}) : \operatorname{tr} X = 0 \}$

with $[X, Y] := XY - YX$.

Then this is a Lie algebra.

Proof. Just need to check $\operatorname{tr}([X, Y]) = 0$

$\operatorname{tr}(XY) = \operatorname{tr}(YX)$ so we're done.

Note that $\mathfrak{sl}(n, \mathbb{C})$ is not closed under usual matrix mult.

Definitions. A (Lie) subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a subspace closed under the bracket.

It is an ideal if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$ (equivalently, vice versa)

If $\mathfrak{g}$ is a cpx Lie alg. and $\mathfrak{h} \subseteq \mathfrak{g}$ is a real subspace, it is a real subalgebra if closed under brackets.

If ~~cases~~ $\mathfrak{g}, \mathfrak{h}$ are Lie algebras, a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism if identically $\phi([X, Y]) = [\phi(X), \phi(Y)]$.

The direct sum of $\mathfrak{g}$ and $\mathfrak{h}$ is the direct sum as vector spaces with

$$[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2], [h_1, h_2]).$$

Can say $\mathfrak{k} = \mathfrak{g} \oplus \mathfrak{h}$ if $\mathfrak{g}, \mathfrak{h}$ are subalgebras with $[g, h] = 0$ for all $g \in \mathfrak{g}$, $h \in \mathfrak{h}$.

7.7. = 8.3

Def. If $\mathfrak{g}$ is a Lie algebra and $X \in \mathfrak{g}$, define a linear map

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$$
$$Y \to [X, Y]$$
$$\mathrm{ad}_X(Y) = [X, Y]$$

the adjoint map or adjoint representation.

Why this notation?

* Instead of $[X,[X,[X,[X,Y]]]]$ write $(\mathrm{ad}_X)^4(Y)$.
* Think of ad as a map $(X \to \mathrm{ad}_X)$
$$\mathfrak{g} \to \mathrm{End}(\mathfrak{g}).$$

Then ad_X is a derivation of the bracket.

$$\mathrm{ad}_X([Y,Z]) = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)].$$

Equivalent to Jacobi!

Proposition. If $\mathfrak{g}$ is a Lie algebra,

$$\mathrm{ad}_{[X,Y]} = \mathrm{ad}_X\, \mathrm{ad}_Y - \mathrm{ad}_Y\, \mathrm{ad}_X = [\mathrm{ad}_X, \mathrm{ad}_Y].$$

Implictly says. $[X, Y]$ behaves like $XY - YX$ even if there is no associative multiplication.

$\mathrm{End}(\mathfrak{g})$ is an associative algebra.

Proof.

Write out what it means:

$$\mathrm{ad}_{[X,Y]}(Z) = \mathrm{ad}_X(\mathrm{ad}_Y(Z)) - \mathrm{ad}_Y(\mathrm{ad}_X(Z)) \quad \forall Z \in \mathfrak{g}$$

i.e. $[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]]$

Jacobi again.

8.4

Def. If $\mathfrak{g}$ is a finite dim Lie algebra w/ basis $X_1, \ldots, X_N$, the structure constants c_{jkl} are determined by

$$[X_j, X_k] = \sum_{l=1}^{N} c_{jkl} X_l .$$

These determine the Lie algebra.

Def. A Lie algebra $\mathfrak{g}$ is irreducible if it has no nontrivial ideals. This is weird.

It is simple if it is irreducible and $\dim \mathfrak{g} \geq 2$.
equivalently, no nontrivial ideals and not abelian.

Prop. The Lie algebra $sl_2(\mathbb{C})$ is simple.
(Recall this is trace zero matrices w/ $[X, Y] = XY - YX$.)

Proof. Write $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

and compute: $[X, Y] = H$, $[H, X] = 2X$ $[H, Y] = -2Y$.

Do by brute force. Suppose $sl_2(\mathbb{C})$ contains an ideal $\mathfrak{h} \ni Z = aX + bH + cY \neq 0$.

If $c \neq 0$, then

$$[X, [X, Z]] = [X, -2bX + cH] = c \cdot -2X .$$

So $X \in \mathfrak{h}$. But now $[X, Y] = H$, etc., etc.

Other cases are the same.

8.5.

Def. If $\mathfrak{g}$ is a Lie algebra its commutator $[\mathfrak{g}, \mathfrak{g}]$ is the vector space generated by $\{[X, Y] : X, Y \in \mathfrak{g}\}$.

Clearly $[\mathfrak{g}, \mathfrak{g}]$ is an ideal.

The upper and lower series.

Define $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_1 = [\mathfrak{g}_0, \mathfrak{g}_0]$, $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1]$ etc.

Each $\mathfrak{g}_i$ an ideal of $\mathfrak{g}_{i-1}$.

These form the derived series of $\mathfrak{g}$; it is solvable if $\mathfrak{g}_j = \{0\}$ for some j.

Also, set $\mathfrak{g}^0 = \mathfrak{g}$, and $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$.

By construction $\mathfrak{g}^i \supseteq \mathfrak{g}_i$, for all i.

This is the upper central series, and $\mathfrak{g}$ is nilpotent if $\mathfrak{g}^j = \{0\}$ for some j.
$(\mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \cdots)$

By construction, nilpotent Lie algebras are also solvable.

Proposition. Let $\mathfrak{g} = \left\{ \begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix} \right\} \subseteq M_3(\mathbb{R})$.

Then $\mathfrak{g}$ is nilpotent.

Proof. Write $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $[X, Y] = Z$, $[X, Z] = [Y, Z] = 0$

So $\mathfrak{g}^1 = \mathfrak{g}_1 = \text{span}(Z)$, more commutators finish it off

Proposition. Let $\mathfrak{g} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subseteq M_2(\mathbb{C})$.

This is a Lie algebra, solvable but not nilpotent.

Proof. Exercise. Interesting part. Write $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then $[H, X] = 2X$, so

$$(\mathrm{ad}_H)^n X = 2^n X, \text{ hence } X \in \mathfrak{g}^j \text{ for all } j.$$

The Lie algebra of a Lie group.

Def. Let G be a matrix Lie group. Its Lie algebra $\mathfrak{g}$ is $\{X : e^{tX} \in G \text{ for all } t \in \mathbb{R}\}$.

i.e. $X \in \mathfrak{g} \iff$ the 1PS generated by X is in $\mathfrak{g}$.

Later: (*) $\mathfrak{g}$ is the tangent space to G at the identity
(soon) the bracket.

(9.1)

Prop. Let G be a matrix Lie group, with $X \in \mathfrak{g}$. Then $e^X \in G_0$ (is in the identity component of G).

Proof. $e^{tX} \in G$ for all t by def., and letting t vary from 0 to 1 defines a continuous path from I to e^X.

(9.2)
8.7. Let G be a matrix Lie group w/ Lie algebra $\mathfrak{g}$.
For all $X, Y \in \mathfrak{g}$ the following are true:

(1) $AXA^{-1} \in \mathfrak{g}$ for $A \in G$
(2) $sX \in \mathfrak{g}$ for all $s \in \mathbb{R}$.
(3) $X + Y \in \mathfrak{g}$

Prove
(4) $XY - YX \in \mathfrak{g}$.

So: This makes $\mathfrak{g}$ into a Lie algebra w/ bracket $[X, Y] = XY - YX$

Proof. (1)

$$e^{t(AXA^{-1})} = A e^{tX} A^{-1} \in G \text{ for all } t. \quad (\text{If } e^{tX} \text{ is})$$

(2) $$e^{t(sX)} = e^{(ts)X}$$

(3) Recall $$e^{t(X+Y)} = \lim_{m \to \infty} \left(e^{tX/m} e^{tY/m} \right)$$

The RHS is in G for all m. Since G is closed, so is the limit.

(4) By the product rule,

$$\frac{d}{dt}\left(e^{tX} Y e^{-tX} \right) = e^{tX} Y \frac{d}{dt}\left(e^{-tX}\right) + \frac{d}{dt}\left(e^{tX} Y\right) e^{-tX}$$

$$= e^{tX} Y \frac{d}{dt}\left(e^{-tX}\right) + \frac{d}{dt}\left(e^{tX}\right) Y e^{-tX}$$

~~e^{tX}~~ Evaluate at $t = 0$:

$$Y \cdot (-X) + XY = [X, Y].$$

8.8 (9.3) We know:

* $e^{tX} Y e^{-tX} \in \mathfrak{g}$ for all t (by 1)

* $\mathfrak{g}$ is topologically closed, so

$$XY - YX = \lim_{h \to 0} \frac{e^{hX} Y e^{-hX} - Y}{h} \quad \text{is in } \mathfrak{g}.$$

By construction we have that $\mathfrak{g}$ is a real vector space.
It might be a complex one, in which case we say
G is complex.

Prop. If G is commutative, then so is $\mathfrak{g}$.
(Later: Converse is true if G is connected.)

Proof. Check that

$$[X, Y] = \frac{d}{dt}\left(\frac{d}{ds} e^{tX} e^{sY} e^{-tX}\Big|_{s=0}\right)\Big|_{t=0}.$$

Basically same as above.

If X, Y commute, get rid of the e^{tX} (and so the t).

8.9 = 9.4 Examples.

In general: write $gl(n)$ = Lie algebra of $GL(n)$
$sl(n)$ " " $SL(n)$

etc.

Prop. $gl_n(\mathbb{C}) = M_n(\mathbb{C})$
$gl_n(\mathbb{R}) = M_n(\mathbb{R})$
$sl_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) : \text{tr } X = 0\}$.

Proof.
$gl_n(\mathbb{C}) : \{X \in M_n(\mathbb{C}) : e^{tX} \in$ ~~gl~~ $GL_n(\mathbb{C})$ for all $t\}$

That's everything!
Same for $\mathbb{R}$. (Also: If e^{tX} is real (for all t), so is $X = \frac{d(e^{tX})}{dt}\Big|_{t=0}$.

$sl_n(\mathbb{C}) = \{$same, with $\det(e^{tX}) = 1$ for all $t\}$

~~Equivalent~~
But $\det(e^{tX}) = e^{\text{trace}(tX)}$
$= e^{t\,\text{trace}(X)}$

So equivalent to demanding $\text{tr}(X) = 0$.

Prop. $u(n) = \{X \in M_n(\mathbb{C}) : X^* = -X\}$
$su(n) = \{X \in M_n(\mathbb{C}) : X^* = -X,\ \text{tr}(X) = 0\}$
$o(n) = \{X \in M_n(\mathbb{R}) : X^T = -X\}$
$so(n) = o(n)$.

Proof: For $U(n)$: M unitary $\leftrightarrow M^* = M^{-1}$.

e^{tX} is unitary $\leftrightarrow (e^{tX})^* = (e^{tX})^{-1}$

$\leftrightarrow e^{tX^*} = e^{-tX}$.

$\leftrightarrow e^{t(X^* + X)} = I$.

This holds for all t simultaneously iff $X^* + X = 0$.

$\Leftarrow$ is obvious, and $\Rightarrow$ is true because we can take logs of both sides whenever t is small.

$o(n)$: same thing, and note that

$$\det(e^{tX}) = 1 \leftrightarrow \operatorname{trace}(X) = 0.$$

Why $so(n) = o(n)$? If $X^T = -X$, trace is automatically 0.

A 1PS is (tautologically) connected, and so must lie in the identity component of G.

So all we care about $O(n)$ is its identity component $SO(n)$.

Ex. Define the Heisenberg group $\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$.

Check: It's a group.

9.6. H's Lie algebra is $\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$.

Why? If X is as above,

$$\exp(tX) = I + \underbrace{tX + \frac{t^2X^2}{2} + \frac{t^3X^3}{6} + \cdots}_{\text{all upper triangular.}}$$

Ex. Check that $su(2)$ is spanned by

$$E_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad E_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad E_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$[E_i, E_{i+1 \pmod 3}] = E_{i+2 \pmod 3}.$$

Now $so(3)$ is spanned by

$$F_1 = \begin{pmatrix} & & \\ & & 1 \\ & -1 & \end{pmatrix} \quad F_2 = \begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix} \quad F_3 = \begin{pmatrix} & -1 & \\ 1 & & \\ & & \end{pmatrix}$$

with the same relations.

So $su(2) \cong so(3)$.

We had a 2-1 surjection $SU(2) \to SO(3)$ so we expected this.

They have neighborhoods of the identity with the same group and topological structure,

9.7

Symplectic groups: Write $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

$$Sp(n) = \{ A : -\Omega A^T \Omega = A^{-1}, \quad \underbrace{A \in U(n)}_{A^* = A^{-1}} \}$$

Then $sp(n) = \{ X \in M_{2n}(\mathbb{C}) : \Omega X^T \Omega = X, \quad X^* = -X \}$.

Verify this above. Think of X as $\log A$.

Can also write $sp(n) = \{ X \in M_n(\mathbb{H}) : X^* = -X \}$.

Theorem. Let G, H be matrix Lie groups w/ algs. $\mathfrak{g}, \mathfrak{h}$.
Suppose $\Phi : G \to H$ is a Lie group homomorphism.
Then there exists a unique real-linear map $\phi : \mathfrak{g} \to \mathfrak{h}$
with

(0) $\Phi(e^X) = e^{\phi(X)}$ for all X, and

(1) $\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}$ for $X \in \mathfrak{g}$, $A \in G$.

(2) $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for $X, Y \in \mathfrak{g}$.

(3) $\phi(X) = \frac{d}{dt} \Phi(e^{tX}) \Big|_{t=0}$ for all $X \in \mathfrak{g}$.

Moreover, the map is functorial: To $id : G \to G$ is associated $id : \mathfrak{g} \to \mathfrak{g}$;

the map associated to $G \xrightarrow{\Phi} H \xrightarrow{\Psi} K$ is $\psi \circ \phi$.

$$\mathfrak{g} \xrightarrow{\phi} \mathfrak{h} \xrightarrow{\psi} \mathfrak{k}$$

9.8

Proof. The point is that Φ must map any 1PS of G to a 1PS of H.

If $X \in \mathfrak{g}$, e^{tX} is a 1PS of G (by def.)

$\Phi(e^{tX})$ is a 1PS of H
(since it is a cts homomorphism)

$\Phi(e^{tX}) = e^{tZ}$ for some Z
(structure thm on 1PS's).

So define $\phi(X) = Z$. This is a map $\mathfrak{g} \to \mathfrak{h}$

Satisfies $\Phi(e^{X}) = e^{\phi(X)}$ by plugging in $t=1$.

Linearity of ϕ:

If $\Phi(e^{tX}) = e^{tZ}$ $\forall t$, then $\Phi(e^{tsX}) = e^{tsZ}$ so
$\phi(sX) = s\phi(X)$.

$$
\begin{aligned}
e^{t\phi(X+Y)} &= \Phi\left(e^{t(X+Y)}\right) \\
&= \Phi\left(\lim_{m\to\infty}\left(e^{\frac{tX}{m}} e^{\frac{tY}{m}}\right)^m\right) && \text{Lie product formula} \\
&= \lim_{m\to\infty}\left(\Phi(e^{tX/m})\,\Phi(e^{tY/m})\right)^m && \Phi \text{ is continuous and a homomorphism} \\
&= \lim_{m\to\infty}\left(e^{t\phi(X)/m}\, e^{t\phi(Y)/m}\right)^m \\
&= e^{t(\phi(X)+\phi(Y))}
\end{aligned}
$$

Differentiate at $t=0$: $\phi(X+Y) = \phi(X) + \phi(Y)$.

9.9

Uniqueness. If also $\Phi(e^{X}) = e^{\phi_1(X)}$ for some ϕ_1,

$$e^{t\phi(X)} = e^{t\phi_1(X)} = \Phi(e^{tX})$$

Differentiate at $t=0$: $\phi(X) = \phi_1(X)$.

Property (1). If $A \in G$,

$$e^{t\phi(AXA^{-1})} = e^{\phi(tAXA^{-1})} = \Phi(e^{tAXA^{-1}})$$

$$= \Phi(A)\,\Phi(e^{tX})\,\Phi(A)^{-1}$$

$$= \Phi(A)\, e^{t\phi(X)}\, \Phi(A)^{-1}.$$

Take the derivative at $t=0 \Rightarrow \phi(AXA^{-1}) = \Phi(A)\,\phi(X)\,\Phi(A)^{-1}$

Property (2). If $X, Y \in \mathfrak{g}$,

$$\phi([X,Y]) = \phi\left(\frac{d}{dt} e^{tX} Y e^{-tX}\Big|_{t=0}\right)$$ (how we defined the Lie bracket) (alternative def)

$$= \frac{d}{dt}\phi(e^{tX} Y e^{-tX})\Big|_{t=0}$$ (derivatives commute w/ linear trans.)

$$= \frac{d}{dt}\left(\Phi(e^{tX})\,\phi(Y)\,\Phi(e^{-tX})\right)\Big|_{t=0}$$ (by (1))

$$= \frac{d}{dt}\left(e^{t\phi(X)}\,\phi(Y)\,e^{-t\phi(X)}\right)\Big|_{t=0}$$

$$= [\phi(X), \phi(Y)].$$

Property 3.

$$\frac{d}{dt}\Phi(e^{tX})\Big|_{t=0} = \frac{d}{dt} e^{t\phi(X)}\Big|_{t=0} = \phi(X).$$

9.10

Functoriality. Let $\Lambda = \Phi \circ \Psi$

Then $\Lambda(e^{+x}) = \Phi(\Psi(e^{+x})) = \Phi(e^{+\psi(x)})$

$= e^{+\phi(\psi(x))}$

So that the map associated to λ is $\phi \circ \psi$.