Exercise Set 1 – Arithmetic Geometry, Frank Thorne (thorne@math.sc.edu)

Due Monday, February 3, 2016

Instructions. Either do all of 1-4, or do 5.

(1) Let V denote the circle $x^2 + y^2 = 1$, $P = (\frac{3}{5}, \frac{4}{5}) \in V(\mathbb{Q})$, and L the line $x = 0$. Let $\phi : V \to L$ denote the map which associates to any $Q \in P$ the intersection of the lines L and PQ.

(a) Prove that, possibly apart from finitely many points, ϕ defines a bijection between $V(K)$ and $L(K)$ for any subfield of $\mathbb{R}$. What points of L correspond to the Pythagorean triples $5^2 + 12^2 = 13^2$ and $8^2 + 15^2 = 17^2$?

(b) Prove that both ϕ and its inverse are both rational functions.

(c) If your earlier bijection excluded the point P itself, define 'the line between P and P' to be the tangent line to V at P and prove that this extends the bijection to P and one point on L which you previously omitted.

(2) Now consider the version of the 'Pythagorean triples parametrization' presented in lecture: let V denote the circle $x^2 + y^2 = 1$, $P = (0, 1) \in V(\mathbb{Q})$, and L the line $x = 0$. Let ϕ be as before.

(a) Write down ϕ and its inverse explicitly as rational functions (e.g., quotients of polynomials in the coefficients).

(b) Now let V' denote the circle $X^2 + Y^2 = Z^2$ in the projective plane $\mathbb{P}^2$. Does it contain any points not on V?

(c) Considering ϕ now as a map to the projective line $\mathbb{P}^1$, write down ϕ and its inverse as *polynomial* functions in the coefficients. Are they defined for all points of V and $\mathbb{P}^1$, or are there finitely many exceptions?

(3) Let p be a prime. Prove the following elementary properties of p-adic valuations which were stated in class. (Here x and y denote arbitrary rational numbers.)

(a) $v_p(xy) = v_p(x)v_p(y)$,

(b) $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$,

(c) $v_p(x + y) = \min\{v_p(x), v_p(y)\}$ if $v_p(x) \neq v_p(y)$.

(4) Prove, using a p-adic valuation argument, that the conic $x^2 + y^2 = p$ has no rational solutions for any $p \equiv 3 \pmod 4$.

(5) Let V be a projective plane smooth conic defined over a field K of characteristic not equal to 2. Assume that V has a K-rational point P. Generalize the 'stereographic projection' argument to exhibit mutually inverse morphisms $\phi : V \to \mathbb{P}^1$ and $\phi^{-1} : \mathbb{P}^1 \to V$, the defining polynomials of which are defined over K.

This proves that V is isomorphic to $\mathbb{P}^1$ over K.