Problem Set 1 – Arithmetic Geometry, Frank Thorne (thorne@math.sc.edu)

Due Friday, September 20, 2024

(1) Adapting the solution to the Gauss circle problem, compute an asymptotic for the number of lattice points within the ellipse $x^2 + 5y^2 = N$. Obtain an explicit bound on the error term without any O-notation.

(2) Recall that *Minkowski's second theorem* states that the successive minima λ_i of a complete lattice Λ in $\mathbb{R}^n$ satisfy

$$\frac{2^n}{n!} \operatorname{Covol}(\Lambda) \leq \lambda_1 \lambda_2 \cdots \lambda_n \cdot \operatorname{Vol}(B(0,1)) \leq 2^n \operatorname{Covol}(\Lambda).$$

Prove either of these two inequalities, either as stated, or with any other constant depending only on n. (Aim for a short and easy solution, rather than the best possible constant.)

(3) Let α be a root of $f(x) := x^3 - 4x - 1$, and write $K := \mathbb{Q}(\alpha)$.

(a) Verify that K has ring of integers $\mathbb{Z}[\alpha]$ and discriminant 229, and is totally real.

(b) Explicitly write down the Minkowski embedding of $\mathcal{O}_K$ into $\mathbb{R}^3$. Use a calculator or computer to write everything down in terms of decimal approximations, to at least three decimal places.

(c) If $1, \alpha, \alpha^2$ were vectors whose lengths are the successive minima, would this be consistent with Minkowski's second theorem?

(d) (Optional) You may wish to try to compute the successive minima, provably or not. This is in general a *highly* nontrivial problem, and you may be interested to poke around the Internet to see what you can learn about this.

(4) Suppose, *contrary to fact*, that the Riemann zeta function was absolutely bounded by some constant M in the region

$$\{z \in \mathbb{C} \ : \ \Re(z) \geq \frac{1}{2}, \ \ |z - 1| > \frac{1}{10}\}.$$

Moreover, define the k-divisor function $d_k(n)$ to be the number of ways to write n as a product of k natural numbers.

(a) Prove the identity

$$\sum_n d_k(n)^s = \zeta(s)^k,$$

valid when $\Re(s) = 1$.

(b) By Perron's formula we have

$$\sum_{n<X} d_k(n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s)^k X^s \frac{ds}{s},$$

whenever X is a positive real number, not an integer. By shifting a portion of the contour left of the line $\Re(s) = 1$, obtain an asymptotic formula for $\sum_{n<X} d_k(n)$ with a power saving error term. You should give *some* description of all the constants in front of your main terms, although it need not be a simplified one.

(Answer this relative to our counterfactual assumption.)

(c) Now, suppose instead of our counterfactual bound, we have a bound

$$|\zeta(\sigma + it)| \ll (1 + |t|)^\alpha$$

for some $\alpha > 0$. Repeat the previous question, obtaining an error term which is presumably worse, but still saves a power of X.

(Note: we can take any $\alpha > \frac{1}{4}$ by the so-called 'convexity bound', any $\alpha > \frac{32}{205}$ by more intricate work of Huxley, and any $\alpha > 0$ if the Riemann Hypothesis is true.)