Exercise Set 1 – Arithmetic Geometry, Frank Thorne (thorne@math.sc.edu)

Due Friday, January 22, 2016

(1) Let V denote the circle $x^2 + y^2 = 1$, $P = (\frac{3}{5}, \frac{4}{5}) \in V(\mathbb{Q})$, and L the line $x = 0$. Let $\phi : V \to L$ denote the map which associates to any $Q \in P$ the intersection of the lines L and PQ.

(a) Prove that, possibly apart from finitely many points, ϕ defines a bijection between $V(K)$ and $L(K)$ for any subfield of $\mathbb{R}$. What points of L correspond to the Pythagorean triples $5^2 + 12^2 = 13^2$ and $8^2 + 15^2 = 17^2$?

(b) Prove that both ϕ and its inverse are both rational functions.

(c) If your earlier bijection excluded the point P itself, define 'the line between P and P' to be the tangent line to V at P and prove that this extends the bijection to P and one point on L which you previously omitted.

(2) Let V now denote the ellipse $ax^2 + by^2 = 1$, and assume that V has a rational point (x_0, y_0) with $x_0 \neq 0$. (This will not be true for all choices of a and b.) Repeat the previous exercise.

If you wish, you may skip writing up the first problem since this one subsumes it. But I don't recommend this. If you find these problems easy, I encourage you to attempt the 'Problems' instead.

(3) Let p be a prime. Prove the following elementary properties of p-adic valuations which were stated in class. (Here x and y denote arbitrary rational numbers.)

(a) $v_p(xy) = v_p(x)v_p(y)$,

(b) $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$,

(c) $v_p(x + y) = \min\{v_p(x), v_p(y)\}$ if $v_p(x) \neq v_p(y)$.

(4) Prove, using a p-adic valuation argument, that the conic $x^2 + y^2 = p$ has no rational solutions for any $p \equiv 3 \pmod 4$.