Exercise Set 2 – Arithmetic Geometry, Frank Thorne (thorne@math.sc.edu)

Due Friday, January 29, 2016

(1) Prove the projective case of the *fundamental theorem of algebra*: given a homogeneous polynomial in 2 variables of degree d, then it factors (over $\mathbb{C}$) as a product of linear factors. Moreover, the roots of these linear factors are uniquely defined in $\mathbb{P}^1$.

(Recall that $[\alpha : \beta]$ is the unique root in $\mathbb{P}^1$ of the linear polynomial $\beta x - \alpha y$.)

You may of course rely on the usual fundamental theorem of algebra in your proof. Do not assume anything about your polynomial, except that it has at least one nonzero coefficient.

(2) Let E be the elliptic curve given by the equation

$$y^2 = x^3 + Ax + B$$

for some complex numbers A and B.

(a) Compute the homogeneous form of this equation, and show the homogeneous form has exactly one additional point (the 'point at infinity') which does not appear in the 'affine patch' above.

(b) The *line at infinity* L in $\mathbb{P}^2$ is given by the equation $z = 0$. Compute the intersection points of L and E together with their multiplicities. Check that your conclusions agree with Bezout's theorem.

(3) Again let E be the elliptic curve given by the equation

$$y^2 = x^3 + Ax + B$$

for some complex numbers A and B. Prove that the following conditions are all equivalent.

(i) E is smooth. (Recall that you must homogenize the curve, write E as $V(f)$, and check that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ do not simultaneously vanish on any point of E. (They are allowed to vanish on points not on E.)

(ii) The cubic $x^3 + Ax + B$ has distinct roots.

(iii) The discriminant $\Delta = -16(4A^3 + 27B^2)$ is nonzero.

(4) Find at least ten pairs of rational numbers (x, y) satisfying the equation

$$y^2 = x^3 + 17.$$