Arithmetic of Invertible Polynomials and Mirror Symmetry

Speaker: Marco Aldi

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The speaker started out by talking about D-modules. This notion means the following. Let $K[X] := K[x_1, \ldots, x_n]$. Then, for each i there are operators x_i – multiplication by x_i, and ∂_i – differentiation with respect to x_i. These operators generate a subalgebra of $\mathrm{End}_K(K[X])$, called the *Weyl algebra* $A_n = A_n(K)$. (Possibly easy question: Do they generate all of $\mathrm{End}_K(K[X])$? Presumably not, although I did not look for a proof.)

Using the fact that $[x_i, \partial_j]$ is $-\delta_{ij}$ times the identity, it is also possible to formally describe A_n by means of generators and relations.

Then the category of D-modules, is by definition, equal to the set of A_n-modules. (Apparently D is supposed to stand for differentials.)

The speaker then described three applications: to the correspondence between regular, holonomic D-modules and constructible sheaves; to Beilinson-Bernstein localization; and to Malgrange's theorem (that differential operators with constant coefficients have Green's functions). Unfortunately, he didn't really explain any of this terminology or the relevance of this D-module theory.

The speaker then described another set of D-modules. Given a polynomial $W \in K[x]$, we have the module $K[X]e^W$. For each element $f(X) \cdot e^W$, x_i and ∂_i act again by differentiation; since the result is always a polynomial times e^W one can think of this as a different module structure on $K[X]$. The *Milnor ring* is then $K[X]/dW$. I got confused by the discussion of how one applies these constructions, but the constructions themselves (and how to maniuplate them) were not difficult to understand.

Finally (during the pretalk), he mentioned that D-modules have applications to Dwork's work on the Weil conjectures. I did not really understand how, but this is fascinating to me! If I understood correctly (I'm not sure that I did), this theory can often be used to effectively and efficiently compute the Hasse-Weil zeta functions.

His main talk was actually more down to earth. Consider the variety $Q : x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0$, together with an action of $G := \mathbb{Z}/5 \times \mathbb{Z}/5$. The first factor acts by multiplying each coordinate by the same root of unity. The second factor acts by multiplying the various coordinates by different (particular) roots of unity. One can compute something called the 'orbifold Euler characteristic', and also find a 'mirror dual' with respect to G^T — which turns out to be $\mathbb{Z}/5 \times \mathbb{Z}/5 \times \mathbb{Z}/5$ with a related action. These turn out to be mirror duals (the explanation was brief, and I didn't really understand it), but it turns out that the construction can be achieved via some fairly elementary matrix computations. Although I did not follow all the details, the computations carried out struck me as interesting and very manageable.