Miniscule varieties and mirror symmetry

Speaker: Sergey Galkin

February 1, 2016

It was an unusual talk. The speaker's boardwork left something to be desired, and I really didn't understand what he proved or what the motivation was. But the talk was intermittently fascinating, and periodically shed great light on some technical topics.

A *flag* in a vector space V of dimension n is a sequence of vector spaces $W_0 \subseteq W_1 \cdots \subseteq W_n$, where each W_i is of dimension i and each dimension is represented. (I believe that some people don't require the latter condition, and use the terminology *full flag* when this is true. Anyway, apparently the set of such form a variety X, called the *flag variety*.

Why is this actually a variety? I asked the speaker this, and apparently there is not a simple answer. (Indeed, even explaining why the Grassmannians are varieties is not so easy.) But in lieu of a direct answer, he explained a relationship to the group action. There is a transitive action of $G = \mathrm{GL}(n)$ on the flag variety (i.e. on the set of all flags). Visibly, the stabilizer G_x of any point x is (conjugate to) an upper triangular matrix, i.e. it is a *Borel subgroup*. Anyway, we have $X = G/G_x$. This is certainly true as sets, but highlights yet another reason why it is interesting to develop a good theory of quotients in algebraic geometry: if we can impose a natural variety stucture on G/G_x, then we can simply say that X inherits it.

Here is another technical construction that I have probably heard before (perhaps from Nick Addington, over a beer) that I understood for the first time. Consider a *quadric surface*, given by the vanishing of a quadratic polynomial in $\mathbb{P}^3$. How many lines are on it?

Consider the *Segre embedding* $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. The lines on $\mathbb{P}^1 \times \mathbb{P}^1$ are just the lines on either $\mathbb{P}^1$, and the image in $\mathbb{P}^3$ is cut out by a simple determinantal condition. Anyway, after a change of variables (if I understood correctly!), *any quadric surface* is isomorphic to this one, and so we have an essentially immediate answer.

In the main talk there was a lot of fascinating discussion of the combinatorics of Lie groups and Lie algebras, which I wish I had understood better. But playing a part in the proof was a reasonably elementary combinatorial construction. Consider the category of *posets*, where the morphisms consist of set maps that preserve inequalities when they exist. For any posets P and Q, $\mathrm{Hom}(P, Q)$ is also a poset: we have $f \geq g$ if $f(x) \geq g(x)$ for all $x \in P$. We therefore obtain a variety of interesting functors, and relevant for the speaker's talk was the functor $\mathrm{Hom}(-, 2)$.

(Here 2 is the poset with two objects, one of which is greater than the other. The initial object 0 is the empty set, and the final object 1 is the singleton.)

Anyway, suppose you start with 1 (say), or any other poset, and you apply this functor to it a bunch of times? A fascinating, easily understood question, and the speaker showed off a few cool-looking examples.