# On a constant related to the Prouhet-Tarry-Escott Problem

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The Prouhet-Tarry-Escott Problem asks the following: Can we find multisets of integers $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ with $\sum_i x_i^e = \sum_i y_i^e$ for each $e$ between 1 and $k$?

The most one can hope for is if $k = n - 1$, in which case the solution is called 'ideal'. (If $k = n$, it follows from a lemma proved by the speaker that the multisets coincide.)

Given an ideal solution, it can be proved (and indeed it was proved by the speaker during the talk) that given an ideal solution we have the identity

$$\prod_{i=1}^{n}(z - x_i) - \prod_{i=1}^{n}(z - y_i) = C,$$

and one can ask questions about $C$. The speaker (if I understood correctly) proved that if $n = 9$ then we must have $2^9 \mid C$. This improves on the previous result $2^7 \mid C$, and indeed *applies* this result, and proceeds by studying the Newton polygons of the two products in the polynomial above. One can say much about these two Newton polygons: since all the roots are integral, they each have to go through ten integer lattice points, and all but the bottom coefficient must be the same (and hence have the same 2-adic valuation).

The proof was presented via slides. I confess that I'm not generally able to keep track of this flavor of slide talk, where the proof is elementary but requires keeping track of a lot of different notation. (This is no fault of the speaker; others, including her advisor Michael Filaseta, do this as well and I generally have difficulty there as well. I would guess that Michael, among others, has a very keen ability to keep track of all of the notation when this flavor of argument is presented quickly. I usually prefer to have some 'machinery' to hang the proof on.)

The talk raises a number of questions. The speaker concentrated on what is known about the constants $C$ for $1 \leq n \leq 12$. The most obvious question (to me) is, are there infinitely many ideal solutions? (One should of course demand nontriviality – and also primitivity: if you multiply all the $x_i$ and $y_i$ by a fixed constant you again get a solution.) Are there solutions for every $n$? If not, are there solutions for infinitely many $n$?

It seems like this kind of question ought to have some connections to additive combinatorics, and I wonder it is possible to study the problem from that angle. (For example, and this is baseless speculation, can one characterize solutions, or give some sort of necessary condition, in terms of higher dimensional almost-arithmetic progressions?) Or, are there connections with Faltings' theorem? The question asks for integral points on an algebraic variety, although the number of variables is roughly double the number of equations. I have no idea, but it seems like a question that makes fertile ground for further inquiry.