Duality in commutative algebra

Speaker: Sean Sather-Wagstaff

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The speaker gave a very nice talk, indeed my favorite commutative algebra talk of all of those that I have seen. The principle was the following: if you want to study an object, study the maps involved: into it, out of it, etc. The speaker focused on 'out of it' today (i.e., the functor $\mathrm{Hom}(-, \mathbb{C})$.

His motivating examples were outside commutative algebra. One was functional analysis. If you have a Hilbert space, then it is common to study its functionals, and this leads to results such as the Riesz representation theorem. Another was algebraic topology: for example, the fundamental group consists of equivalence classes of maps $S^1 \to X$. (Note that here the X is on the other side.)

One example I find interesting (although probably less related to the speaker's work) comes from Fourier analysis, namely the *Pontryagin dual*. For example, in Fourier analysis, $\mathbb{Z}$ is dual to $\mathbb{R}/\mathbb{Z}$. Is there any good way to 'explain' or 'understand' this? (The speaker seemed to imply that the answer was usually no in commutative algebra, but perhaps this is well known in Fourier analysis).

Let R be a noetherian commutative ring with 1, and consider the category of R-modules. Define a duality operation by taking $M* = \mathrm{Hom}_R(M, N)$ for some fixed N. What is a good choice? What properties might we expct this to satisfy?

One property is 'zeroness', i.e., that $M = 0$ if and only if $M^* = 0$. This usually fails. Similarly one could ask whether $M^{**} \simeq M$, or whether duality preserves short exact sequences. These tend to fail, at least without some further hypotheses on the modules M.

The speaker defined a finitely generated module M to be *totally C-reflextive* if these latter properties are all true, and indeed a strengthening of these properties is true. Studying these, one obtains conditions like $\mathrm{Ext}^i_R(C, C) = 0$. (The Ext functor is defined as follows: from a short exact sequence of modules $0 \to M_1 \to M_2 \to M_3 \to 0$ we obtain $0 \to M_3^* \to M_2^* \to M_1^*$, and Ext^i is what you keep filling in to get a long exact sequence. In particular, if $\mathrm{Ext}^1(M, C) = 0$ always then short exact sequences are preserved.)

A *semidualizing module* is one C that satisfies the following condition: (1) The natural map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism (we obtain elements of $\mathrm{Hom}_R(C, C)$ by multiplication by r; are these all distinct, and the only ones?); and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geq 1$. C is *dualizing* if in addition it has finite injective dimension, i.e. there is an injective resolution of bounded length. The main question was: given a ring R, how many (non-isomorphic) SDM's does R have? At this point I had gotten rather bogged down in the details, but the speaker raised some interesting technical details. He proved that there are finitely many for any local ring R, and the actual computations (for which the answers are often single-digit numbers) seemed quite nontrivial.