Hyperkähler geometry

Speaker: Justin Sawon

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The speaker's talk was given using Beamer slides, involved a lot of technical machinery, and I understood very little of it. But in his very interesting pretalk, he described some interesting geometric questions, and indirectly gave me a little bit better intuition for differential geometry.

His basic question was: *how to do geometry over the quaternions*? Let's take it for granted that one knows what a real differentiable manifold is. What should a *complex structure* be? My first instinct would be to define it in direct analogy, but Sawon described a somewhat more highbrow approach: we can define a map 'multiplication by i', and then demand that the differential $Df_p : T_p \to T_{f(p)}$ commute with this map. For example, given a map $f : \mathbb{R}^2 \to \mathbb{R}^2$, this amounts to the claim that the two matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

commute, which is equivalent to the Cauchy-Riemann equations.

For the quaternions, it turns out the only solutions are affine ($z \to az + b$), yielding a poor theory of quaterionic structures. I was wondering during the talk: is it possible to prove this in a more elementary way? For example, for what functions f from and to the quaternions does the limit $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exist? (Here h ranges over quaternions, not real numbers.) By change of variables, one could presumably assume that $z = f(0) = 0$, and ask that $\lim_{h \to 0} \frac{f(h)}{h}$ exists. But what does the fraction even mean?! We have to divide 'on the left' or 'on the right' — so I can now appreciate the need for a more complicated definition.

A *hypercomplex* manifold is a smooth manifold with three complex structures I, J, K satisfying $IJ = K$, etc. Apparently these do exist. It is *hyperkähler* if .... I could copy the definition here, but it wasn't too enlightening to this nonexpert.

But here is one bit of algebraic formalism which is quite simple, but which represents a new way of thinking for me. Suppose you have a 2-form ω. Then the speaker claimed that this induces a map ω_p (for every point p in the manifold M) from the tangent space T_pM to the cotangent space T_p^*M. I did not at all see how you would get such a map! But once the speaker said what it was – namely, the map sending u to $\omega(u, -)$ – it was quite clear. My (very incomplete) understanding is that this way of thinking is fundamental to the categorical perspective. And, indeed, I saw such a transformation in an undergraduate computer science class (!) where the professor explained how to rewrite functions of multiple variables in terms of functions of one variable.

I learned one more interesting fact from his talk (unrelated to anything above or to the main point he was making): If S is a surface, then the Hilbert scheme of two points on S can be constructed as follows: First, take $S \times S$ and quotient by the involution; then, blow up the resulting 4-fold on the diagonal.