BUBBLES OF CONGRUENT PRIMES

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Abstract. In [15], Shiu proved that if a and q are arbitrary coprime integers, then there exist arbitrarily long strings of consecutive primes which are all congruent to a modulo q. We generalize Shiu's theorem to imaginary quadratic fields, where we prove the existence of "bubbles" containing arbitrarily many primes which are all, up to units, congruent to a modulo q.

1. Introduction and Statement of Results

In 1997, Shiu [15] proved the remarkable result that if a, q, and k are arbitrary integers with $(a, q) = 1$, there exists a string of k consecutive primes

$$p_{n+1} \equiv p_{n+2} \equiv \cdots \equiv p_{n+k} \equiv a \pmod{q}.$$

(Here p_n denotes the nth prime.) Furthermore, for k sufficiently large in terms of q, these primes can be chosen to satisfy the bound[1]

$$\frac{1}{\phi(q)}\left(\frac{\log\log p_{n+1} \log\log\log\log p_{n+1}}{(\log\log\log p_{n+1})^2}\right)^{1/\phi(q)} \ll k, \tag{1.1}$$

uniformly in q.

In this paper we consider an analogous question for imaginary quadratic fields. If K is such a field, then the ring of integers $\mathcal{O}_K$ forms a lattice in $\mathbb{C}$, and the primes of $\mathcal{O}_K$ can be naturally visualized as lattice points. In this setting one may ask whether there are clumps of primes, all of which lie in a fixed arithmetic progression. We prove that this is indeed the case, up to multiplication by units:

Theorem 1.1. *Suppose K is an imaginary quadratic field, k is a positive integer, and a and q are elements of $\mathcal{O}_K$ with $q \neq 2$ and $(a, q) = 1$. Then there exists a "bubble"*

$$B(r, x_0) := \{x \in \mathbb{C} : |x - x_0| < r\} \tag{1.2}$$

with at least k primes, such that all the primes in this bubble are congruent to ua modulo q for units $u \in \mathcal{O}_K$. Furthermore, for k sufficiently large in terms of q (and

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[1] In Shiu's statement of his results, the initial $1/\phi(q)$ in (1.1) and the requirement that k be large are omitted, and the implied constant in (1.1) is allowed to depend on q. A careful reading of his proof shows that the dependence on q may be controlled as stated.

K), x_0 can be chosen to satisfy

$$\frac{1}{\phi_K(q)}\left(\frac{\log\log|x_0|\log\log\log\log|x_0|}{(\log\log\log|x_0|)^2}\right)^{\omega_K/h_K\phi_K(q)} \ll k. \tag{1.3}$$

The implied constant is absolute.

Here ω_K denotes the number of units in $\mathcal{O}_K$, h_K is the class number of K, and $\phi_K(q) := |(\mathcal{O}_K/(q))^\times|$.

Remarks. The unit u will not necessarily be the same for each prime in the bubble (1.2). It would be desirable to obtain a version of Theorem 1.1 where each prime is congruent to a modulo q, without the ambiguity involving units. Unfortunately, this ambiguity appears to be unavoidable given our methods of proof.

The restriction that $q \neq 2$ is not severe; to obtain prime bubbles modulo 2 we may take (for example) $q = 4$. For the reason behind this restriction, see Lemma 3.1.

Example. Let $K = \mathbb{Q}(i)$, $q = 5+i$, and $a = 1$. A computer search reveals that the ball of radius $\sqrt{7.5}$ centered at $2+17i$ contains three primes, all of which are congruent to ± 1 or $\pm i$ modulo q. Similarly the ball of radius $\sqrt{23.5}$ centered at $59+779i$ contains six primes, all of which are congruent to ± 1 or $\pm i$. Theorem 1.1 establishes the existence of infinitely many such balls, with $\omega_K/\phi_K(q) = 1/3$.

The proof of Theorem 1.1 is an adaptation of Shiu's original proof [15], and in particular uses the Maier matrix method. (See the survey article of Granville [7] for an interesting overview of the method and related results.) In our proof, we will construct "Maier matrices" containing certain elements of $\mathcal{O}_K$, and our construction will force the majority of the primes in these matrices to be congruent to ua modulo q. Some combinatorial geometry will then allow us to deduce Theorem 1.1.

To use Maier's method we will require a result on the distribution of primes in certain arithmetic progressions. This result is an analogue of a theorem of Gallagher ([6]; see also [11], Lemma 2), who proved that the primes of $\mathbb{Z}$ are reasonably well distributed in arithmetic progressions modulo q, for reasonably large moduli q which meet certain conditions on the associated Dirichlet L-functions. We will prove the following analogue of Gallagher's theorem for $\mathcal{O}_K$:

Theorem 1.2. *Let K be an imaginary quadratic field. Suppose that $q \in \mathcal{O}_K$ is a modulus for which none of the Hecke L-functions modulo q have a zero in the region*

$$\sigma > 1 - C_1/\log[(\mathbb{N}q)(|t|+1)], \tag{1.4}$$

for a fixed constant C_1. Suppose further that q is not u, $2u$, or $\frac{-3\pm\sqrt{-3}}{2}u$ for any unit u of $\mathcal{O}_K$. Then for $D \geq 0$ we have

$$\pi(2x;q,a) - \pi(x;q,a) = (\omega_K + o_{x,D}(1))\frac{x}{h_K\phi_K(q)\log x},$$

uniformly in q for $(a,q)=1$, $\mathbb{N}q \geq |\Delta_K|$, and $x \geq \mathbb{N}q^D$.

Here $\pi(x; q, a)$ denotes the number of principal prime ideals $\mathfrak{p}$ of norm $\leq x$, such that $p \equiv a \pmod{q}$ for any generator p of $\mathfrak{p}$, and $o_{x,D}(1)$ denotes an error term bounded above by any $\epsilon > 0$, provided both x and D are chosen sufficiently large. We remark that the condition $\mathbb{N}q \geq |\Delta_K|$ is required only if the $o_{x,D}(1)$ term is to be independent of K.

We will further prove in Proposition 5.1 that the zero-free region (1.4) holds for a suitably large (infinite) set of moduli q.

Generally speaking, the results of this paper indicate that the Maier matrix method "works" for imaginary quadratic fields (at least), and we believe that it would not be difficult to prove the existence of various irregularities in the distribution of the primes of $\mathcal{O}_K$, in analogy with results for $\mathbb{Z}$ obtained by Maier [12], Granville and Soundararajan [8], and others. We have not, however, undertaken this task here.

The outline of the paper is as follows. In Section 3 we briefly overview some background material on Hecke characters and Hecke L-functions, and we discuss how the units of $\mathcal{O}_K$ affect our analysis. We will then give the proof of Theorem 1.2. In Section 4 we prove the previously mentioned result in combinatorial geometry which will allow us to deduce the existence of a prime bubble from our Maier matrix construction. In Section 5 we will prove several additional lemmas, and finally we give the proof of Theorem 1.1 in Section 6.

Setup and notation. We make the following assumptions throughout (some of which were mentioned earlier). K is an imaginary quadratic field with a fixed embedding $K \to \mathbb{C}$, and we will write h_K for the class number of K and $\omega = \omega_K \in \{2, 4, 6\}$ for the number of units in K. Any K-dependence of implicit constants occuring in our results will be explicitly noted.

We will write $\mathfrak{q} = (q)$ throughout, and where it does not lead to ambiguity we will refer to $\mathfrak{q}$ and q interchangeably. We assume that $\mathfrak{q}$ has been chosen so that the units of $\mathcal{O}_K$ all represent distinct residue classes mod $\mathfrak{q}$, and we will prove in Lemma 3.1 that this only excludes three choices for $\mathfrak{q}$. We further assume that the units do not represent all reduced residue classes modulo $\mathfrak{q}$; if this happens then Theorem 1.1 is trivial.

As K will be fixed, we will simply write $\phi(q)$ (or $\phi(\mathfrak{q})$) for $\phi_K(q) := |(\mathcal{O}_K/(q))^*|$. We will also write $h_\mathfrak{q}$ for $h_K\phi(q)/\omega$, the size of the ray class group.

It will be convenient to define congruences on ideals. For a principal ideal $\mathfrak{b}$ of $\mathcal{O}_K$ and elements a and q of $\mathcal{O}_K$, we will say that $\mathfrak{b} \equiv a \pmod{q}$ if $b \equiv a \pmod{q}$ for any b for which $\mathfrak{b} = (b)$. This does not determine a uniquely, and indeed if $\mathfrak{b} \equiv a$ then $\mathfrak{b} \equiv ua$ for any unit $u \in \mathcal{O}_K$. Equivalently, we see that $\mathfrak{b} \equiv a \pmod{q}$ if $\mathfrak{b}$ and (a) represent the same class in the ray class group $H^{(q)}$.

For any nonprincipal ideal $\mathfrak{b}$ we will say that $\mathfrak{b} \not\equiv a \pmod{q}$ for each a.

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3. Proof of Theorem 1.2

We will need to do some analysis involving Hecke L-functions. For the sake of completeness we give a brief overview of the definitions and terminology here.

We would like to consider elements of $\mathcal{O}_K$ and their distribution in the quotient group $\mathcal{O}_K/\mathfrak{q}$. However, it will prove much easier to work with ideals. In place of $\mathcal{O}_K/\mathfrak{q}$ we consider the *ray class group* modulo $\mathfrak{q}$

$$H^{\mathfrak{q}} := J^{\mathfrak{q}}/P^{\mathfrak{q}}, \tag{3.1}$$

where $J^{\mathfrak{q}}$ is the group of all fractional ideals coprime to $\mathfrak{q}$, and $P^{\mathfrak{q}}$ is the group of principal fractional ideals $(a) = (b)(c)^{-1}$ with $b, c \in \mathcal{O}_K$ and $b \equiv c \equiv 1 \mod \mathfrak{q}$. If we write $J_1^{\mathfrak{q}}$ for the group of principal fractional ideals coprime to $\mathfrak{q}$, then $J_1^{\mathfrak{q}}/P^{\mathfrak{q}}$ is in one-to-one correspondence with the set of sets of reduced residue classes modulo $\mathfrak{q}$

$$\{ua : (a, \mathfrak{q}) = 1, u \in \mathcal{O}_K^{\times}\}, \tag{3.2}$$

where a is a fixed in each set and u ranges over all units of $\mathcal{O}_K$. The proof of Theorem 1.1 will exhibit bubbles of prime elements p, such that the ideals (p) all lie in a fixed class in $J_1^{\mathfrak{q}}/P^{\mathfrak{q}}$.

With a few exceptions, the size of $H^{\mathfrak{q}}$ is given by the following simple formula:

Lemma 3.1. *Suppose that $\mathfrak{q} \neq (2), \left(\frac{-3\pm\sqrt{-3}}{2}\right)$ and that $\phi(\mathfrak{q}) > 1$. Then we have*

$$|H^{\mathfrak{q}}| = h_K \phi(\mathfrak{q})/\omega, \tag{3.3}$$

where h_K is the class number of K, and $\omega \in \{2, 4, 6\}$ denotes the number of units of $\mathcal{O}_K$.

We recall that we will write $h_{\mathfrak{q}} = h_K \phi(\mathfrak{q})/\omega$ throughout, as this quantity occurs throughout our analysis.

Proof. We first note that $J^{\mathfrak{q}}/J_1^{\mathfrak{q}}$ is isomorphic to the usual class group, so it suffices to show that there are $\phi(\mathfrak{q})/\omega$ sets counted in (3.2). And as long as $u - 1 \notin \mathfrak{q}$ for any unit $u \neq 1$ of $\mathcal{O}_K$, we see that a cannot be congruent to ua modulo q for any unit u and reduced residue a. Therefore, the residues ua in (3.2) lie in distinct classes for each unit u, and so we obtain the formula (3.3). Thus, it is enough to check that $u - 1 \notin \mathfrak{q}$ for $u \in \{-1, \pm\sqrt{-1}, \frac{\pm1\pm\sqrt{-3}}{2}\}$:

If $u = -1$, then (3.3) holds unless $2 \in \mathfrak{q}$, in which case either $\phi(\mathfrak{q}) = 1$ or $\mathfrak{q} = (2)$.

If $u = \pm\sqrt{-1}$, then (3.3) holds unless $\mathfrak{q} = (1 + i)$, in which case $\phi(\mathfrak{q}) = 1$.

If $u = \frac{1\pm\sqrt{-3}}{2}$, then $u - 1$ is also a unit and is not contained in any proper ideal.

If $u = \frac{-1\pm\sqrt{-3}}{2}$, then (3.3) holds unless $\frac{-3\pm\sqrt{-3}}{2} \in \mathfrak{q}$, in which case $\mathfrak{q} = \left(\frac{-3\pm\sqrt{-3}}{2}\right)$. □

Remark. In the case where $\mathfrak{q} = \left(\frac{-3\pm\sqrt{-3}}{2}\right)$ and $K = \mathbb{Q}(\sqrt{-3})$, we observe that the units of $\mathcal{O}_K$ cover all reduced residue classes mod $\mathfrak{q}$, so that Theorem 1.1 follows trivially.

From the group $H^{\mathfrak{q}}$ we obtain *Hecke characters* χ of K by lifting any character χ of $H^{\mathfrak{q}}$ to $J^{\mathfrak{q}}$ in the obvious way, and setting $\chi(\mathfrak{a}) = 0$ for any a not coprime to q. Throughout, we will only consider Hecke characters obtained in this fashion. (See, however, Chapter VII.6 of [13] (for example) for a much more general discussion.) For a Hecke character χ, the associated *Hecke L-function* is defined by the equation

$$L(s,\chi) := \sum_{\mathfrak{a}} \chi(\mathfrak{a})(\mathbb{N}\mathfrak{a})^{-s}, \tag{3.4}$$

where $\mathfrak{a}$ runs over all integral ideals of $\mathcal{O}_K$. The proof of Theorem 1.2 will then follow from the following estimate:

Proposition 3.2. *Assume that* $\mathfrak{q}$ *is a modulus such that none of the Hecke L-functions modulo* $\mathfrak{q}$ *have a zero in the region (1.4). Then if* $\max(\exp(\log^{1/2} x), \Delta_K) \leq \mathbb{N}\mathfrak{q} \leq x^b$ *for a fixed constant* $b > 0$*, then we have for a fixed constant* a

$$\sum_{\chi} \Big| \sum_{\mathbb{N}\mathfrak{p}\in[x,2x]} \chi(\mathfrak{p}) \log(\mathbb{N}\mathfrak{p}) \Big| \ll x \exp\Big(-a \frac{\log x}{\log \mathbb{N}\mathfrak{q}} \Big). \tag{3.5}$$

The first sum is over all nonprincipal characters modulo $\mathfrak{q}$*, and the implied constant is absolute.*

We remark that with additional care we expect to be able to prove a similar result for an arbitrary number field K.

Before proving Proposition 3.2 we will use it to derive Theorem 1.2:

Proof of Theorem 1.2. By the orthogonality relations, we have

$$\sum_{\substack{\mathbb{N}\mathfrak{p}\in[x,2x] \\ \mathfrak{p}\equiv a \mod q}} \log(\mathbb{N}\mathfrak{p}) = \frac{1}{h_{\mathfrak{q}}} \sum_{\mathbb{N}\mathfrak{p}\in[x,2x]} \sum_{\chi \pmod{\mathfrak{q}}} \bar{\chi}(a)\chi(\mathfrak{p}) \log(\mathbb{N}\mathfrak{p})$$

$$= \frac{1}{h_{\mathfrak{q}}} \sum_{\mathbb{N}\mathfrak{p}\in[x,2x]} \log(\mathbb{N}\mathfrak{p}) + O\bigg(\frac{1}{h_{\mathfrak{q}}} \sum_{\chi\neq\chi_0} \Big| \sum_{\mathbb{N}\mathfrak{p}\in[x,2x]} \chi(\mathfrak{p}) \log(\mathbb{N}\mathfrak{p}) \Big| \bigg),$$

and for $x \leq \exp((\log \mathbb{N}q)^2)$, the result now follows from the prime ideal theorem and Proposition 3.2.

For the range $x > \exp((\log \mathbb{N}q)^2)$, one proof can be given as follows: Taking $T = \exp((\log x)^{3/4})$ in the proof of Proposition 3.2, we see that the quantity in (3.5) is $\ll x\exp(-a(\log x)^{1/4})$, and this suffices for our result. □

To prove Proposition 3.2 we will closely follow Gallagher [6]. Gallagher proves a similar result for Dirichlet L-functions, but with an additional sum over moduli q. He deduces his result from a log-free zero-density estimate for these L-functions, and in our case the appropriate zero-density estimate has been proved by Fogels [5]:

Proposition 3.3 (Fogels). *We have for any $\mathfrak{q} \in \mathcal{O}_K$ and any $T \geq \Delta_K \mathbb{N}\mathfrak{q}$*

$$\sum_{\chi} N_{\chi}(\alpha, T) \leq T^{c(1-\alpha)}. \tag{3.6}$$

Here $N_{\chi}(\alpha, T)$ denotes the number of zeroes $\rho = \beta + it$ of $L(s, \chi)$ with $\alpha < \beta < 1$ and $|t| < T$, χ ranges over all characters modulo $\mathfrak{q}$, Δ_K is the discriminant of K, and c is (for quadratic fields) an absolute constant.

Proof of Proposition 3.2. At the outset, we will choose $T = (\mathbb{N}\mathfrak{q})^2 \leq x^{1/2c}$, which is an acceptable choice in all of our estimates.

By standard analytic techniques (see (5.53) and (5.65) of [9]), we have

$$\sum_{\mathbb{N}\mathfrak{a} \in [x, 2x]} \chi(\mathfrak{a}) \Lambda(\mathfrak{a}) = \delta_{\chi} x - \sum_{\rho} \frac{(2x)^{\rho} - x^{\rho}}{\rho} + O\Big(\frac{x \log^2 x}{T}\Big), \tag{3.7}$$

where δ_{χ} is 1 or 0 according to whether χ is principal or not, $\Lambda(\mathfrak{a}) := \log(\mathbb{N}\mathfrak{p})$ if $\mathfrak{a}$ is a power of some prime $\mathfrak{p}$ and 0 otherwise, and ρ ranges over all the zeroes $\rho = \beta + it$ of $L(s, \chi)$ in the critical strip with $|t| < T$.

We observe that for each $\rho = \beta + it$,

$$\frac{(2x)^{\rho} - x^{\rho}}{\rho} \ll x^{\beta}.$$

The terms where $\mathfrak{a}$ is a prime power (but not a prime) contribute $\ll x^{1/2}$ to the sum (3.7) and so may be absorbed into the error term for $T \leq x^{1/2}$. Therefore, for nonprincipal χ we see that

$$\sum_{\mathbb{N}\mathfrak{p} \in [x, 2x]} \chi(\mathfrak{p}) \log(\mathbb{N}\mathfrak{p}) \ll \sum_{\rho} x^{\beta} + \frac{x \log^2 x}{T}.$$

Therefore,

$$\sum_{\chi \neq \chi_0} \Big| \sum_{\mathbb{N}\mathfrak{p} \in [x, 2x]} \chi(\mathfrak{p}) \log(\mathbb{N}\mathfrak{p}) \Big| \ll \sum_{\chi \neq \chi_0} \sum_{\rho} x^{\beta} + \frac{x \log^2 x (\mathbb{N}\mathfrak{q})}{T}.$$

The sum over χ and ρ on the right is

$$-\int_0^1 x^{\sigma} d_{\sigma}\Big(\sum_{\chi \neq \chi_0} N_{\chi}(\sigma, T)\Big) = -x^{\sigma}\Big(\sum_{\chi \neq \chi_0} N_{\chi}(\sigma, T)\Big)\Big|_0^1 + \int_0^1 x^{\sigma} \log x \Big(\sum_{\chi \neq \chi_0} N_{\chi}(\sigma, T)\Big) d\sigma. \tag{3.8}$$

The first term of (3.8) is ([9], Theorem 5.8)

$$\sum_{\chi \neq \chi_0} N_{\chi}(0, T) \ll T \mathbb{N}\mathfrak{q} \log(T \mathbb{N}\mathfrak{q}).$$

Using the zero-free region (1.4) and Proposition 3.3, we see that the second term of (3.8) is

$$\ll \int_0^{1-C_1/\log[(\mathbb{N}\mathfrak{q})(T+1)]} (x^\sigma \log x) T^{c(1-\sigma)} d\sigma.$$

Evaluating the integral above and recalling that $T \leq x^{1/2c}$, this second term is

$$\ll x \exp\left(-\frac{C_1}{2} \frac{\log x}{\log[(\mathbb{N}\mathfrak{q})(T+1)]}\right).$$

We conclude from all these estimates that

$$\sum_\chi{}' \left| \sum_{\mathbb{N}\mathfrak{p} \in [x,2x]} \chi(\mathfrak{p}) \log(\mathbb{N}\mathfrak{p}) \right| \ll$$

$$\frac{(x \log^2 x)\mathbb{N}\mathfrak{q}}{T} + T\mathbb{N}\mathfrak{q} \log(T\mathbb{N}\mathfrak{q}) + x \exp\left(-\frac{C_1}{2} \frac{\log x}{\log[(\mathbb{N}\mathfrak{q})(T+1)]}\right).$$

With the choice $T = (\mathbb{N}\mathfrak{q})^2$ and the hypothesis that $\max(\exp(\log^{1/2} x), |\Delta_K|) \leq \mathbb{N}\mathfrak{q} \leq \min(x^{1/4c}, x^{1/4})$, we obtain the proposition. □

4. Bubbles of Good and Bad Points

We will require a combinatorial argument to prove the existence of bubbles of "good" primes. This part of the proof has nothing to do with primes in particular, so we formulate it as a general proposition in combinatorial geometry.

Throughout, we will write $C(r)$ for the circle centered at the origin of radius r. When we speak of a circle containing points, we will mean that these points are in the interior of the circle and not on the circle itself.

Proposition 4.1. *Suppose the plane contains some number of "good" and "bad" points, such $C(1)$ contains g good points and $C(3)$ contains b bad points. Then there exists some circle in the plane containing $> g/(2b+12)$ good points and no bad points.*

In our application to the proof of Theorem 1.1, b and g will be large with $b = o(g)$. The construction will be scaled and translated to appropriate regions of the complex plane.

We gratefully acknowledge a contribution from Bob Hough, who suggested ideas that have allowed us to improve Proposition 4.1 and simplify the proof.

We shall require the existence of a so-called Delaunay triangulation:

Lemma 4.2. *Let $\mathcal{P}$ be a set of points in the plane, not all collinear. Then there exists a triangulation (called a Delaunay triangulation) of $\mathcal{P}$, such that no point of $\mathcal{P}$ is inside the circumcircle of any triangle.*

See, e.g., Chapter 9 of [3] for a proof of this. In the (unlikely) case where all points of $\mathcal{P}$ are collinear, the proof of Proposition 4.1 is trivial.

We will also need the following lemma:

Lemma 4.3. *Let $\mathcal{P}$ be a set of N points in the plane, not all collinear, and let k denote the number of points in $\mathcal{P}$ that lie on the boundary of the convex hull of $\mathcal{P}$. Then any triangulation of $\mathcal{P}$ has $2n - 2 - k$ triangles.*

This follows easily from Euler's formula; see Theorem 9.1 of [3] for details.

Proof of Proposition 4.1. The proof is by geometric construction. Define a set of vertices V, consisting of all bad points of distance less than 3 from the origin, as well as a regular 7-gon centered at the origin, so that the distance from each vertex to the origin is 2.

Construct the Delaunay triangulation T of V, and let $\mathcal{C}$ be the set of circumcircles of all triangles in T. Then let $\mathcal{C}' \subseteq \mathcal{C}$ be the the subset of those circles which intersect the interior of the unit circle. By construction, no circle in $\mathcal{C}'$ contains any point of V, and the circles in $\mathcal{C}'$ cover the interior of $C(1)$, with the exception of any bad points.

We claim that every circle in $\mathcal{C}'$ is contained in the interior of $C(3)$. Supposing for now that this happens, we know that the circles in $\mathcal{C}'$ do not contain any bad points, including any which may lie on or outside $C(3)$. These circles do contain all of the good points in $C(1)$, and it follows that one such circle contains $\geq g/|\mathcal{C}'|$ good points. Lemma 4.3 implies that $|\mathcal{C}'| < 2(b+7) - 2$, and this establishes the proposition.

It remains to prove our claim, and the proof is by contradiction. In particular, suppose that C is some circle in $\mathcal{C}'$ not contained in $C(3)$; then C or its interior will contain points P_1 on $C(1)$ and P_3 on $C(3)$. Furthermore, we may take these points to be on the ray from the origin towards the center of C. We also easily check that C must contain the circle having $\overline{P_1P_3}$ as its diameter.

For some point Q of the 7-gon, the angle between $\overrightarrow{OQ}$ and $\overrightarrow{OP_3}$ is at most $\pi/7$ and in particular is less than $\pi/6$. We check that the distance between Q and the midpoint of $\overline{P_1P_3}$ is then less than 1, which implies that Q is contained in the interior of C, our contradiction.

□

5. Additional Lemmas

We introduce the notation

$$\mathcal{P}(y, q, \mathfrak{p}_0) := q \prod_{\mathbb{N}\mathfrak{p} \leq y; \mathfrak{p} \neq \mathfrak{p}_0} \mathfrak{p}. \tag{5.1}$$

Our first result is that for a sufficient number of moduli $\mathcal{P}(y, q, \mathfrak{p}_0)$, the associated Hecke L-functions have a zero-free region of the form required by Theorem 1.2.

Lemma 5.1. *For all sufficiently large x there exist an integer y and a prime $\mathfrak{p}_0$ with $x < \mathbb{N}\mathcal{P}(y, q, \mathfrak{p}_0) \ll x \log^3 x$ and $\mathbb{N}\mathfrak{p}_0 \gg \log y$, such that none of the Hecke L-functions modulo $\mathcal{P}(y, q, \mathfrak{p}_0)$ have a zero in the region*

$$1 \geq \Re s > 1 - \frac{C_2}{\log[(\mathbb{N}\mathcal{P}(y, q, \mathfrak{p}_0))(|t| + 1)]} \tag{5.2}$$

for a fixed constant C_2.

Here "sufficiently large" depends on K. We could easily control this K-dependence here, but it would be more difficult in Lemma 5.3.

The lemma and its proof are the direct analogues of Theorem 1 of [15]. We will require the following zero-free region for Hecke L-functions, due to Fogels [4]:

Lemma 5.2 (Fogels)**.** *Assume that $\mathfrak{a}$ is an ideal of $\mathcal{O}_K$ with $|\Delta_K \mathbb{N}\mathfrak{a}|$ sufficiently large. Then for a fixed absolute constant C, at most one of the Hecke L-functions $L(s,\chi)$ modulo $\mathfrak{a}$ has any zeroes in the region*

$$\sigma \geq 1 - \frac{C}{\log[|\Delta_K|\mathbb{N}\mathfrak{a}(|t|+1)]} \geq \frac{3}{4}. \tag{5.3}$$

Furthermore, if such an $L(s,\chi)$ exists, then it has at most one zero β in the region (5.3), which is necessarily real, and

$$\beta < 1 - (|\Delta_K|\mathbb{N}\mathfrak{a})^{-4}. \tag{5.4}$$

Remark. The above results in fact hold for an arbitrary number field K. In this case C depends on the degree of K, and the exponent -4 in (5.4) should be replaced with $-2[K:\mathbb{Q}]$.

Proof of Lemma 5.1. Consider the product

$$\mathcal{P}'(y,q) := q \prod_{\mathbb{N}\mathfrak{p} \leq y} \mathfrak{p}, \tag{5.5}$$

and suppose that an exceptional character mod $\mathcal{P}'(y,q)$ exists; i.e., suppose that there exists a character χ_1 mod $\mathcal{P}'(y,q)$ whose L-function has a real zero β in the range

$$1 \geq \beta \geq 1 - \frac{C}{\log(|\Delta_K|\mathbb{N}\mathcal{P}'(y,q))}. \tag{5.6}$$

Write χ_1' (mod $\mathcal{P}''$) for the primitive character inducing χ_1, so that $\mathcal{P}''|\mathcal{P}'(y,q)$. Then comparing (5.6) with (5.4) we see[2] that $\mathbb{N}\mathcal{P}'' \gg \frac{1}{|\Delta_K|}(\log \mathbb{N}\mathcal{P}'(y,q))^{1/4}$. We thus see that for sufficiently large y (in terms of q), $\mathcal{P}''$ will have a prime divisor $\mathfrak{p}_0$ satisfying $\mathfrak{p}_0 \gg \log(\mathbb{N}\mathcal{P}'') \gg \log\log(\mathbb{N}\mathcal{P}'(y,q)) \gg \log y$.

We claim that there can be no character χ_2 modulo $\mathcal{P}(y,q,\mathfrak{p}_0)$ whose L-function has a real zero in the region

$$\beta' > 1 - \frac{C}{2\log(|\Delta_K|\mathbb{N}\mathcal{P}(y,q,\mathfrak{p}_0))}. \tag{5.7}$$

[2] If $|\Delta_K|$ is small it might be the case that $\mathcal{P}''$ is of too small norm to apply (5.4). For each such K we may choose a fixed ideal $\mathfrak{b}$ of sufficiently large norm, and write χ_1'' for the character modulo $\mathfrak{b}\mathcal{P}''$ induced by χ_1'. The associated L-function will have a zero at the same spot, and we conclude that $\mathbb{N}(\mathfrak{b}\mathcal{P}'') \gg \frac{1}{|\Delta_K|}(\log \mathbb{N}\mathcal{P}'(y,q))^{1/4}$. As $\mathfrak{b}$ is fixed for each K, this implies that $\mathbb{N}\mathcal{P}'' \gg \frac{1}{|\Delta_K|}(\log \mathbb{N}\mathcal{P}'(y,q))^{1/4}$ as well.

Assuming this for now, we obtain the region (5.2) for $\mathcal{P}(y, q, \mathfrak{p}_0)$ with $C_2 = C/4$, provided that y is large enough so that $\mathbb{N}\mathcal{P}(y, q, \mathfrak{p}_0) \geq |\Delta_K|$. To prove our claim, suppose such a χ_2 exists. Then β' will be in the region (5.6), and as χ_2 and χ_1' induce different characters modulo $\mathcal{P}'(y, q)$, β and β' will be zeroes to distinct L-functions modulo $\mathcal{P}'(y, q)$ in the region (5.3), contradicting Lemma 5.2.

If no exceptional character mod $\mathcal{P}'(y, q)$ exists, we choose $\mathfrak{p}_0$ to be any prime divisor of $\mathcal{P}'(y, q)$ of norm $\geq \log y$. We again take $C_2 = C/4$ and see that (for large y) no L-function modulo $\mathcal{P}(y, q, \mathfrak{p}_0)$ will have a zero in the region (5.7).

To conclude, we must show that we can find a $\mathcal{P}(y, q, \mathfrak{p}_0)$ in each range $x < \mathbb{N}\mathcal{P}(y, q, \mathfrak{p}_0) \ll x \log^3 x$. In quadratic fields there can exist at most two distinct primes of the same norm. For a fixed large y, let $y' > y$ be minimal so that $\mathcal{P}(y', q) \neq \mathcal{P}(y, q)$. Then $\mathbb{N}\mathcal{P}(y', q)/\mathbb{N}\mathcal{P}(y, q) \leq (y')^2 = (1+o(1)) \log^2(\mathbb{N}\mathcal{P}(y', q))$, so for any large x we can find y with $2x \log x < \mathbb{N}\mathcal{P}(y, q) < 3x \log^3 x$. Removing a prime $\mathfrak{p}_0$ from our product we see that necessarily $\mathbb{N}\mathfrak{p}_0 \leq y = (1 + o(1)) \log x$ and so $x < \mathbb{N}\mathcal{P}(y, q, \mathfrak{p}_0) \ll x \log^3 x$, as desired. □

Lemma 5.3. *Let $\mathcal{S}(x)$ denote the number of ideals of norm $\leq x$ whose prime (ideal) factors are all $\equiv 1 \pmod{\mathfrak{q}}$. Then*

$$\mathcal{S}(x) = (C_\mathfrak{q} + o_\mathfrak{q}(1))x(\log x)^{-1+1/h_\mathfrak{q}}, \tag{5.8}$$

where

$$C_\mathfrak{q} := \frac{1}{\Gamma(1/h_\mathfrak{q})} \lim_{s \to 1^+} \left[(s-1)^{1/h_\mathfrak{q}} \prod_{\mathfrak{p} \equiv 1 \pmod{\mathfrak{q}}} \left(1 - \frac{1}{(\mathbb{N}\mathfrak{p})^{-s}}\right)^{-1} \right]. \tag{5.9}$$

Proof. This is a generalization of Landau's work on sums of two squares, and also of Lemma 3 of [15]. Write

$$F(s) := \prod_{\mathfrak{p} \equiv 1 \pmod{\mathfrak{q}}} \left(1 - \frac{1}{(\mathbb{N}\mathfrak{p})^{-s}}\right)^{-1}. \tag{5.10}$$

Then by a Tauberian theorem due to Raikov ([2], Theorem 2.4.1), the asymptotic (5.8) follows if we can write

$$F(s) = \frac{H(s)}{(s-1)^{1/h_\mathfrak{q}}}$$

for a function $H(s)$ which is holomorphic and nonzero in the region $\Re(s) \geq 1$, with

$$C_\mathfrak{q} = \frac{H(1)}{\Gamma(1/h_\mathfrak{q})}.$$

We write

$$\Theta(s) := \frac{\prod_{\chi \pmod{\mathfrak{q}}} L(s, \chi)}{F(s)^{h_\mathfrak{q}}}, \tag{5.11}$$

and computing the Dirichlet series expansion for $\log \Theta(s)$ (exactly as in [15]) we conclude that $\Theta(s)$ is holomorphic for $\Re(s) > \frac{1}{2}$. The product $\prod_{\chi \pmod{\mathfrak{q}}} L(s,\chi)$ has a simple pole at $s = 1$, and is otherwise holomorphic and nonzero in $\Re(s) \geq 1$. The result follows. □

We now need a result from the theory of 'smooth' numbers, i.e., numbers whose prime factors are all sufficiently small. (See, for example, Chapter III.5 of Tenenbaum's book [16] for a general introduction to the theory.) Here we require a result for 'smooth' algebraic integers in K.

Lemma 5.4. *Let $\Psi_K(x,y)$ be the number of ideals of norm $< x$ which are composed only of primes with norm $< y$, and write $u := \log x/\log y$. Then for $1 \leq u \leq \exp(c(\log y)^{3/5-\epsilon})$ (for a certain constant c) we have*

$$\Psi_K(x,y) \ll_K x \log^2 y \exp(-u(\log u + \log\log u + O(1))). \tag{5.12}$$

Proof. This follows immediately by comparing results of de Bruijn [1] and Krause [10]. de Bruijn proved (5.12) for $K = \mathbb{Q}$. For general K, Krause proved an asymptotic formula for $\Psi_K(x,y)$ in terms of the Dickman function, and Krause's result implies in particular that for u in the range specified,

$$\lim_{x,y\to\infty} \frac{\Psi_K(x,y)}{\Psi(x,y)} = \mathrm{res}_{s=1}\zeta_K(s),$$

where $\zeta_K(s)$ denotes the Dedekind zeta function. The lemma then follows immediately. □

6. Proof of Theorem 1.1

We begin by fixing a and $\mathfrak{q} = (q)$; except when noted to the contrary, implied constants in our analysis do not depend on $\mathfrak{q}$. As discussed previously, we assume that the units of $\mathcal{O}_K$ do not represent all the reduced residue classes modulo $\mathfrak{q}$, and that the residue classes represented are all distinct.

We assume that a large absolute constant D is given, as well as an integer x which is sufficiently large in terms of $\mathfrak{q}$ (and K). We then use Lemma 5.1 to choose y and $\mathfrak{p}_0$ such that

$$x^{1/D} < \mathbb{N}\mathcal{P}(y,q,\mathfrak{p}_0) \ll x^{1/D}\log x$$

and such that there is no Hecke L-function modulo $\mathcal{P}(y,q,\mathfrak{p}_0)$ with a zero in the region (5.2). We introduce variables $z < y$ and $t < (yz)^{1/2}$, and define a set of primes $\mathcal{P}$ as follows: If a is not congruent to a unit modulo $\mathfrak{q}$, we define

$$\mathcal{P} := \begin{cases} \{\mathfrak{p} : \mathbb{N}\mathfrak{p} \leq y, \mathfrak{p} \neq \mathfrak{p}_0, \mathfrak{p} \not\equiv 1, a \mod \mathfrak{q})\} \\ \cup \{\mathfrak{p} : t \leq \mathbb{N}\mathfrak{p} \leq y, \mathfrak{p} \neq \mathfrak{p}_0, \mathfrak{p} \equiv 1 \mod \mathfrak{q}\} \\ \cup \{\mathfrak{p} : \mathbb{N}\mathfrak{p} \leq yz/t, \mathfrak{p} \neq \mathfrak{p}_0, \mathfrak{p} \equiv a \mod \mathfrak{q}\}. \end{cases} \tag{6.1}$$

If a is congruent to a unit modulo $\mathfrak{q}$, we define instead

$$\mathcal{P} := \left\{ \begin{array}{l} \{\mathfrak{p} : \mathbb{N}\mathfrak{p} \leq y, \mathfrak{p} \neq \mathfrak{p}_0, \mathfrak{p} \not\equiv 1 \mod \mathfrak{q}\} \\ \cup \{\mathfrak{p} : t \leq \mathbb{N}\mathfrak{p} \leq yz/t, \mathfrak{p} \neq \mathfrak{p}_0, \mathfrak{p} \equiv 1 \mod \mathfrak{q}\}. \end{array} \right. \tag{6.2}$$

The latter definition (6.2) is motivated by simplicity, as it allows us to treat both cases simultaneously. Following Shiu [15], it should be possible to define $\mathcal{P}$ differently in this case, and modestly improve our result for a certain subset of moduli a.

We further define

$$\mathfrak{Q} = (Q) := \mathfrak{q} \prod_{\substack{\mathfrak{p} \in \mathcal{P} \\ \mathfrak{p} \neq \mathfrak{p}_1}} \mathfrak{p}. \tag{6.3}$$

Here $\mathfrak{p}_1$ is any prime ideal with $\log y < \mathbb{N}\mathfrak{p}_1 \leq y$ for which $\mathfrak{Q}$ is then principal. We may then write Q for any generator of $\mathfrak{Q}$.

We see that $\mathfrak{Q} | \mathcal{P}(y, q, \mathfrak{p}_0)$ and $\log(\mathbb{N}Q) \geq \frac{1}{3} \log(\mathbb{N}\mathcal{P}(y, q, \mathfrak{p}_0))$. Lemma 5.1 thus implies that the Hecke L-functions modulo $\mathfrak{Q}$ have no zeroes in the region

$$1 \geq \Re s > 1 - \frac{C_2}{3 \log[(\mathbb{N}Q)(|t| + 1)]}, \tag{6.4}$$

as any such zeroes would induce zeroes of L-functions modulo $\mathcal{P}(y, q, \mathfrak{p}_0)$ at the same point. Therefore Q satisfies the hypothesis of Theorem 1.2 with $C_1 = C_2/3$, so that the primes are well-distributed in arithmetic progressions modulo Q.

Our construction is an adaptation of that of Shiu. In our case, the geometrical argument given in Section 4 requires us to keep track of more "bad" primes than "good". Thus we define "bubbles" B and B' consisting of those elements of $\mathcal{O}_K$ whose norm is less than yz and $9yz$, respectively. We further define Maier matrices M and M', with (i, b) entry equal to the algebraic integer $iQ + b$, where i ranges over all elements of $\mathcal{O}_K$ with norm in $(\mathbb{N}Q^{D-1}, 2\mathbb{N}Q^{D-1})$, and b ranges over elements of B and B' respectively. We regard M naturally as a submatrix of M'.

We define sets

$$S := \{i \in B; (i, Q) = 1; i \equiv ua \mod q \text{ for some } u \in \mathcal{O}_K^\times\} \tag{6.5}$$

and

$$T := \{i \in B'; (i, Q) = 1; i \not\equiv ua \mod q \text{ for any } u \in \mathcal{O}_K^\times\}. \tag{6.6}$$

We will prove that S is much larger than T.

To estimate S, we observe that most elements of S are uniquely determined as elements of the form pn, where p is a prime of norm $> yz/t$ and is congruent to ua for some unit u, and n is a product of primes congruent to 1 modulo q. (There will also be multiples of $\mathfrak{p}_0$ and $\mathfrak{p}_1$, which we ignore.) Subdividing dyadically, we see that

$$|S| \geq \sum_{i=0}^{\lfloor \frac{\log t}{\log 2} \rfloor - 2} \Big(\pi(2^{i+1} yz/t; q, ua) - \pi(2^i yz/t; q, ua) \Big) \mathcal{S}(t/2^{i+1})$$

$$\gg \frac{C_{\mathfrak{q}}}{h_{\mathfrak{q}}} \sum_{i=0}^{\lfloor \frac{\log t}{\log 2} \rfloor - i_0} \left(\frac{yz2^i}{t \log y} \right) \cdot \frac{t}{2^{i+1}} \log(t/2^{i+1})^{-1+1/h_{\mathfrak{q}}}.$$

Here i_0 is a constant, depending on q, such that Lemma 5.3 gives an asymptotic estimate for $x \gg 2^{i_0}$. We now simplify and approximate the sum by the corresponding integral, and conclude that

$$|S| \gg \frac{C_{\mathfrak{q}} yz}{h_{\mathfrak{q}} \log y} \int_0^{\frac{\log t}{\log 2} - i_0} \big(\log t - s \log 2 \big)^{-1+1/h_{\mathfrak{q}}} ds \tag{6.7}$$

$$= \frac{C_{\mathfrak{q}} yz}{(\log 2)(\log y)} \left((\log t)^{1/h_{\mathfrak{q}}} - (i_0 \log 2)^{1/h_{\mathfrak{q}}} \right) \gg \frac{C_{\mathfrak{q}} yz}{\log y} (\log t)^{1/h_{\mathfrak{q}}}.$$

Elements of T come in three types: multiples of $\mathfrak{p}_0$ and $\mathfrak{p}_1$, multiples of a prime of norm greater than y, or products of a unit and elements whose norms are less than t and are congruent to 1 modulo q. We write T', T'', T''' for these subsets of T respectively and we will estimate each in turn. We have $|T'| \ll yz/\log y$ because $\mathbb{N}\mathfrak{p}_0, \mathbb{N}\mathfrak{p}_1 \gg \log y$. For T'', we have that

$$|T''| \leq \sum_{i=0}^{\lceil \frac{\log(9z)}{\log 2} \rceil - i_0} \Big(\pi(2^{i+1}y) - \pi(2^i y) \Big) \mathcal{S}(9z/2^i) + \Big(\pi(9yz) - \pi(yz/2^{i_0}) \Big) \mathcal{S}(9 \cdot 2^{i_0}).$$

$$\ll \sum_{i=0}^{\lceil \frac{\log(9z)}{\log 2} \rceil - i_0} \left(\frac{2^i \omega y}{h_K \log y} \right) \cdot \frac{C_{\mathfrak{q}} z}{2^i} (\log(9z/2^i))^{-1+1/h_{\mathfrak{q}}} + O_q\Big(\frac{yz}{\log y} \Big).$$

In the above, $\pi(x)$ counts the number of prime elements of norm $\leq x$. Estimating in the same way as in (6.7), we conclude that

$$|T''| \ll C_{\mathfrak{q}} \phi(q) \frac{yz(\log z)^{1/h_{\mathfrak{q}}}}{\log y}.$$

To count elements T''' we apply Lemma 5.4. We choose (as in [15])

$$t = \exp\Big(\frac{\log y \log \log \log y}{4 \log \log y} \Big), \tag{6.8}$$

and the lemma implies that

$$|T'''| = \omega \Psi(yz, t) \ll yz(\log t)^2 \exp(-4 \log \log y + o(\log \log y)) \ll \frac{yz}{\log y}.$$

Putting these estimates together we conclude that

$$|T| \ll C_{\mathfrak{q}} \phi(q) \frac{yz(\log z)^{1/h_{\mathfrak{q}}}}{\log y}. \tag{6.9}$$

If y is large in terms of K, then the implied constant does not depend on K.

Write P_1 for the number of primes in M (henceforth "good primes") congruent to ua modulo q for any unit $u \in \mathcal{O}_K$, and write P_2 for the number of primes ("bad primes") in M' not congruent to ua for any u. By Theorem 1.2, P_1 and P_2 are determined by $|S|$ and $|T|$, up to an error term which can be made small by choosing large x and D. We therefore conclude that

$$P_1 \gg C_{\mathfrak{q}} \frac{yz(\log t)^{1/h_{\mathfrak{q}}}}{\log y} \frac{\mathbb{N}Q^D}{\phi(Q)\log(\mathbb{N}Q^D)} \tag{6.10}$$

and

$$P_2 \ll C_{\mathfrak{q}} \phi(q) \frac{yz(\log z)^{1/h_{\mathfrak{q}}}}{\log y} \frac{\mathbb{N}Q^D}{\phi(Q)\log(\mathbb{N}Q^D)}.$$

We will split into two cases and compare numbers of good and bad primes. Throughout, we count all bad primes appearing in M' (which contains M), but only those good primes appearing in M.

In the first case the majority of good primes occur in rows containing at least one bad prime, in which case the proportion of good to bad primes in some such row of M' is $\gg |S|/|T|$. These primes all occur in some circle in $\mathbb{C}$ of radius $3\sqrt{yz}$, and applying Proposition 4.1 we see that this circle contains a subcircle with $\gg |S|/|T|$ good primes and no bad primes, which is our desired bubble of congruent primes. The number of primes in the bubble will be

$$\gg |S|/|T| \gg \frac{1}{\phi(q)} \Big(\frac{\log t}{\log z}\Big)^{1/h_{\mathfrak{q}}}.$$

In the second case, the majority of good primes occur in rows not containing any bad primes. These such rows then constitute bubbles of congruent primes of radius $3\sqrt{yz}$, and at least one will contain $\gg P_1/R$ primes, where R denotes the number of rows, i.e., the number of elements of $\mathcal{O}_K$ with norm in $(\mathbb{N}Q^{D-1}, 2\mathbb{N}Q^{D-1})$. As $\mathcal{O}_K$ forms a lattice in $\mathbb{C}$ we have $R \sim C_K \mathbb{N}Q^{D-1}$ for some constant C_K depending on K. Using (6.10), we see that some row of M will be a bubble containing

$$\gg_K C_{\mathfrak{q}} \frac{yz(\log t)^{1/h_{\mathfrak{q}}}}{\log y} \frac{\mathbb{N}Q}{\phi(Q)\log(\mathbb{N}Q^D)}$$

primes. Now we have

$$\log(\mathbb{N}Q) \ll \sum_{\mathbb{N}\mathfrak{p} \leq y} \log(\mathbb{N}\mathfrak{p}) \ll y,$$

and

$$\frac{\mathbb{N}Q}{\phi(Q)} = \frac{\mathbb{N}\mathfrak{q}}{\phi(\mathfrak{q})} \prod_{\mathfrak{p} \in \mathcal{P}} \Big(1 - \frac{1}{\mathbb{N}\mathfrak{p}}\Big)^{-1} \gg_{\mathfrak{q}} \log y (\log t)^{-1/h_{\mathfrak{q}}}. \tag{6.11}$$

To prove (6.11), one can use a result of Rosen (Theorem 4 of [14], along with the result of Landau cited immediately afterwards). The result is then easily proved, provided that the dependence on $\mathfrak{q}$ (and K) is allowed.

Combining these results, we conclude that this bubble contains $\gg_{\mathfrak{q}} z$ primes. Therefore, our argument produces a bubble of

$$\gg \min\Big(\frac{1}{\phi(q)}\Big(\frac{\log t}{\log z}\Big)^{1/h_{\mathfrak{q}}}, C'_{\mathfrak{q}} z\Big)$$

congruent primes, for a constant $C'_{\mathfrak{q}}$ depending on $\mathfrak{q}$. Our theorem follows by choosing $z = \log\log(\mathbb{N}Q)$.

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